Solve at most 5 of the following problems.

**Pb. 1)** Let \( x_1, \ldots, x_n \) be real numbers. Prove that

1a) \( \sum_{i=1}^{n} x_i^2 = 1 \implies (x_1^2 x_2^2 \cdots x_n^2)^{1/n} \leq 1/n \);

and then deduce that

1b) \( x_i > 0 \) for every \( i \implies (x_1 x_2 \cdots x_n)^{1/n} \leq \frac{x_1 + \cdots x_n}{n} \).

**Pb. 2)** Let \( f, g : \mathcal{D} \to \mathcal{C} \) be holomorphic. Prove that the condition

\[
|f(z)| \leq |g(z)| \quad (z \in \mathcal{D}),
\]

implies the existence of a constant \( c \) such that, \( f = cg \).

**Pb. 3)** Let \( H \) a Hilbert space, and for any \( n \geq 1 \) let \( A_n : H \to H \) be a bounded linear operator. Prove that \( \|A_n\| \to 0 \) if the following condition holds:

\( A_n x_n \to 0 \) strongly for every weakly convergent sequence \( (x_n)_{n \geq 1} \) in \( H \).

**Pb. 4)** Let \( (f_n)_{n \geq 1} \) be a sequence of real continuous functions defined on \([0, 1]\). Prove that the following conditions are equivalent:

a) \( (f_n)_{n \geq 1} \) is equibounded and \( f_n \to 0 \) pointwise;

b) \( \lim_n \int_{[0, 1]} f_n \, d\mu = 0 \) for every Borel measure \( \mu \) which is positive and bounded on \([0, 1]\).

**Pb. 5)** Let \( X \) be a complete metric space and, for each \( 0 \leq \lambda \leq 1 \), let \( T_\lambda : X \to X \) satisfy

\[
d(T_\lambda(x), T_\lambda(y)) \leq \frac{1}{2} d(x, y) \quad (\forall \ x, y, \lambda).
\]

Prove that if \( D \) is a dense subset of \( X \) and for every \( x \in D \)

\[
\lim_{\lambda \to 0} T_\lambda(x) = T_0(x),
\]

then,

\[
\lim_{\lambda \to 0} x_\lambda = x_0,
\]

where \( x_\lambda = T_\lambda(x_\lambda) \).

**Pb. 6)** In \( L^\infty = L^\infty([0, 1]) \), let \( V \) be the set of all characteristic function \( \chi_E \), with \( E \) running in the set of measurable subsets of \([0, 1]\). Is \( V \) compact in \( L^\infty \)? Justify your answer.
Pb. 7) Let $T : \ell_2 \to \ell_2$ be the linear operator defined by

$$T(x) := \left(\frac{x_n}{n}\right)_{n \geq 1},$$

for every $x = (x_n)_{n \geq 1} \in \ell_2$. Prove that $T(\ell_2)$ is dense in $\ell_2$ and differs from $\ell_2$ (that is, $\overline{T(\ell_2)} = \ell_2$ and $T(\ell_2) \neq \ell_2$).

Pb. 8) Let $f, f_n : [0, 1] \to \mathbb{R}$ be continuous functions. Assume that for every sequence $(x_n)_{n \geq 1}$ converging to a point $x \in [0, 1]$, we have

$$\lim_n f_n(x_n) = f(x).$$

Prove that $f_n$ converges uniformly to $f$ in $[0, 1]$.

Pb. 9) Let $f : \mathbb{R} \to ]0, +\infty[$ be continuous. Prove the uniqueness of the solution to the Cauchy problem

$$x' = f(x), \quad x(t_0) = x_0,$$

for every initial data $t_0, x_0$.

Pb. 10) Let $f : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ be of class $C^1$. Let $x$ be the solution to the Cauchy problem

$$x'(t) = f(t, x(t)), \quad x(0) = 0,$$

on $[0, \beta] \subset [0, 1]$. Prove that if $x$ cannot be extended at $\beta$ as a solution, then for every compact subset $K \subset \mathbb{R}^n$ there exists $t_K < \beta$ such that $x(t) \notin K$ for $t_K < t < \beta$. 

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