Solve at most five of the following problems, explaining your answers.

1. Let $f \in C^\infty(\mathbb{R})$.
   (i) Prove that for any $x \in \mathbb{R}$ the Cauchy problem
   \[
   \begin{cases}
   u' = e^{-f(u)^2} \\
   u(0) = x
   \end{cases}
   \]
   admits a unique solution $u(\cdot, x)$ defined on $\mathbb{R}$.
   (ii) Show that $\lim_{t \to -\infty} u(t, x) = -\infty$ and $\lim_{t \to +\infty} u(t, x) = +\infty$.

2. Let
   \[c_0 = \{(x_n)_{n \in \mathbb{N}} \subset \mathbb{R} : \lim_{n \to \infty} x_n = 0\}\]
   and let $u : c_0 \to \mathbb{R}$ be a linear bounded functional. For every $n \in \mathbb{N}$ set $\eta_n = u(e^{(n)})$, where $e^{(n)}_j = 0$ for $j \neq n$ and $e^{(n)}_n = 1$. Prove that $(\eta_n) \in l^1$ and $\|u\| = \sum_{n \in \mathbb{N}} |\eta_n|$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function and for every $x \in \mathbb{R}$ let
   \[f'_+(x) = \lim_{h \to 0^+} \frac{f(x + h) - f(x)}{h}.
   \]
   Show that
   \[\int_a^b f'_+(x)ds = f(b) - f(a)\]
   for any $a, b \in \mathbb{R}$ with $a < b$.

4. Let $(X, d)$ be a complete metric space, $\bar{x} \in X$, $r > 0$ and $D = \{x \in X : d(x, \bar{x}) \leq r\}$. Let $f : D \to X$ satisfying
   \[d(f(x), f(y)) \leq kd(x, y) \text{ for any } x, y \in D\]
   where $k \in (0, 1)$ is a constant. Prove that if $d(\bar{x}, f(\bar{x})) \leq r(1 - k)$ then $f$ admits a unique fixed point.
5. Let $I = (0, 1)$ and let $(f_n)$ be a bounded sequence in $L^p(I)$ ($1 < p < \infty$). Prove that, if $(f_n)$ converges to 0 in $L^1(I)$, then it converges to 0 in $L^r(I)$ for any $r$ with $1 \leq r < p$.

6. Characterize the points of $\mathbb{R}^2$ where the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by
   \[ f(x, y) = \max\{|x|, |y|\} \]
   is differentiable.

7. For any $\alpha \geq 0$ let $X^\alpha$ be the space of all functions $u \in L^2(0, 2\pi)$ whose Fourier coefficients
   \[ u_n = \int_0^{2\pi} e^{-int} u(t)dt \]
satisfy
   \[ \sum_{n \in \mathbb{Z}} |n|^{2\alpha} |u_n|^2 < \infty. \]
   It is known that $X^\alpha$ is a Hilbert space with norm
   \[ \|u\|_\alpha = \left( |u_0|^2 + \sum_{n \in \mathbb{Z}} |n|^{2\alpha} |u_n|^2 \right)^{\frac{1}{2}}. \]

Prove that, if $0 \leq \alpha < \beta$, then the imbedding $X^\beta \hookrightarrow X^\alpha$ is compact.

8. Let $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Find the solution $u \in C^2(B) \cap C^0(\overline{B})$ to the problem:
   \[ \begin{cases} 
   \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0 & \forall (x, y) \in B \\
   u(x, y) = 2x^2 & \forall (x, y) \in \partial B.
   \end{cases} \]

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that $f$ is convex if and only if
   \[ \int_{-\infty}^{+\infty} f(x)[\varphi(x + h) + \varphi(x - h) - 2\varphi(x)]dx \geq 0 \]
   for any $h \in \mathbb{R}$ and for any $\varphi \in C^\infty_0(\mathbb{R})$ with $\varphi \geq 0$.

10. Let $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be the space of all linear maps from $\mathbb{R}^n$ into $\mathbb{R}^n$, let $A : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be a function of class $C^1$ and let $f(x) = A(x)x$ for every $x \in \mathbb{R}^n$. Show that if $A(0)$ is invertible then $f$ is locally invertible in a neighborhood of 0.