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Sector of Functional Analysis and Applications

Entrance Examination for the curricula of Mathematical Analysis and Applied Mathematics
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Solve at most five of the following problems.

1. Let $f \in C(\mathbb{R}^2, \mathbb{R})$ be such that $|f(t, y)| < 1$ for any $(t, y) \in \mathbb{R}^2$. Consider the solution $y$ of the Cauchy problem
\[
\begin{cases}
y''(t) + 3y'(t) + 2y(t) = f(t, y(t)), \\
y(0) = y'(0) = 0.
\end{cases}
\]
Prove that $|y(t)| < 1$ for any $t \geq 0$.
(Hint: Use the methods of variations of constants.)

2. Let $f \in C^1(\mathbb{R}, \mathbb{R})$ be such that $f'(t) > 0$ for any $t \in \mathbb{R}$.
   (i) Prove that the image $f(\mathbb{R})$ is an open interval (possibly unbounded).
   (ii) Assuming that $f'(t) \geq \frac{1}{1 + f(t)^2}$, for any $t \in \mathbb{R}$
        prove that $f(\mathbb{R}) = \mathbb{R}$.

3. Given $E \subseteq [0, 1]$, let $\chi_E$ be the indicatrix function of $E$, defined by
   
   \[
   \chi_E(t) = \begin{cases}
   1 & \text{if } t \in E, \\
   0 & \text{if } t \notin E.
   \end{cases}
   \]

   Let $(E_n)_{n \geq 0}$ be a sequence of measurable subsets of $[0, 1]$ and $E$ a measurable subset of $[0, 1]$ such that $\chi_{E_n} \rightharpoonup \chi_E$ weakly in $L^2([0, 1])$.

   Prove that $\chi_{E_n} \to \chi_E$ strongly in $L^2([0, 1])$.

4. For any $f \in C([0, 1], \mathbb{R})$ set
   
   \[
   Tf(x) = \int_0^1 \min\{x, y\} \cdot f(y)dy.
   \]

   Prove that $T$ is a compact operator from $C([0, 1], \mathbb{R})$ into itself and find its spectrum.

5. Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space and let $f : X \to \mathbb{R}^+$ be a measurable function such that
   
   \[
   \mu \{x \in X : f(x) > t\} > \frac{1}{1 + t}.
   \]
Prove that \( f \) is not integrable.

6. Let \( a = (a_n)_{n \in \mathbb{N}} \) be a sequence of real positive numbers such that \( a_n \to +\infty \) as \( n \to \infty \).
Consider the Hilbert space

\[ \ell^2_a = \left\{ u = (u_n)_{n \in \mathbb{N}} \text{ sequence in } \mathbb{C} \text{ such that } \sum_{n=1}^{\infty} a_n |u_n|^2 < +\infty \right\}, \]

endowed with the scalar product

\[ (u, v) = \sum_{n=1}^{\infty} a_n u_n \overline{v}_n. \]

Prove that the closed unit ball in the Hilbert space

\[ \ell^2 = \left\{ u = (u_n)_{n \in \mathbb{N}} \text{ sequence in } \mathbb{C} \text{ such that } \sum_{n=1}^{\infty} |u_n|^2 < +\infty \right\}, \]

endowed with the usual scalar product, is a compact subset of \( \ell^2_a \).

7. Let \( K = \{ x : [0, T] \to \mathbb{R} \text{ such that } x'(t) = x^2(t) \text{ and } 0 \leq x(T) \leq 1 \} \).
   Prove that \( K \) is a compact subset of \( C([0, T], \mathbb{R}) \).

8. Let \( f \in C^2(\mathbb{R}^2, \mathbb{R}) \) be a function with first and second derivatives globally bounded and such that \( f(x, 0) = f(0, y) = 0 \) for all \( (x, y) \in \mathbb{R}^2 \).
   Prove that there exists \( C > 0 \) such that \( |f(x, y)| \leq C |xy| \) for any \( (x, y) \in \mathbb{R}^2 \).

9. Let \( f \in C^1(\mathbb{R}, \mathbb{R}) \). Consider a not identically zero solution \( y \) of the equation

\[ y''(t) = f(y(t)), \quad t \in [0, 1]. \]

Prove that \( y \) has at most a finite number of zeroes.

10. Consider the system of ordinary differential equations

\[ \begin{cases} x' = -x^2 - xy + x \\ y' = -y^2 - xy + y, \end{cases} \]

with initial conditions \( x(0) = x_0 > 0 \) and \( y(0) = y_0 > 0 \).
   Prove that the solution \( (x(t), y(t)) \) is defined for any \( t \geq 0 \) and that \( x(t) > 0 \) and \( y(t) > 0 \) per for any \( t \geq 0 \).