S.I.S.S.A.
Sector of Functional Analysis and Applications

Entrance examination for curricula in Mathematical Analysis and Applied Mathematics
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Solve at most five of the following problems.

1. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be continuous. Consider the system of differential equations
   \[ x''(t) = f(x(t))x(t), \]
   where $x(\cdot)$ takes values in $\mathbb{R}^3$. Prove that the trajectory of any solution $x(\cdot)$ is contained in the subspace of $\mathbb{R}^3$ generated by the vectors $x(0)$ and $x'(0)$.

2. Given $a \in \mathbb{R}$, consider the $2 \times 2$ system of ordinary differential equations
   \[
   \begin{align*}
   x'(t) &= x(t) + ay(t) \\
y'(t) &= ay(t).
   \end{align*}
   
   Say for which values of $a$ there exists a solution $(x(\cdot), y(\cdot))$ not identically zero such that $(x(t), y(t)) \to (0, 0)$ as $t \to +\infty$.

3. In this problem consider known the completeness in $L^2((0, \pi))$ of the system of functions $\{\cos(kx) : k = 0, 1, 2, \ldots\} \cup \{\sin(kx) : k = 1, 2, \ldots\}$.
   (a) Prove that the closed subspace of $L^2((0, \pi))$ generated by $\{\sin(kx) : k = 1, 2, \ldots\}$ coincides with $L^2(0, \pi)$.
   (b) Prove that the closed subspace of $L^2((-\pi, \pi))$ generated by $\{1\} \cup \{\sin(kx) : k = 1, 2, \ldots\}$ does not coincide with $L^2((-\pi, \pi))$.

4. Let $f \in C^1(\mathbb{R}, \mathbb{R})$ be a function such that $|f(x) - \cos x| \leq 1$ for every $x \in \mathbb{R}$. Prove that all the solutions of the equation $x'(t) = f(x(t))$ are bounded.

5. Say if the Cauchy problem
   \[
   \begin{align*}
x'(t) &= \sqrt[3]{(x(t) - 1)^2} \\
x(0) &= 0.
   \end{align*}
   
   has a unique solution defined on the whole $\mathbb{R}$, and justify the answer.

6. Set
   \[
   B^0 = \{u \in C^0([0, 1]) : \|u\|_{C^0} \leq 1\};
   \]
   \[
   B^1 = \{u \in C^1([0, 1]) : \|u\|_{C^1} \leq 1\}.
   \]
Answer to the following questions and justify the answers.

(a) Is $B^0$ a closed subset of $L^1([0, 1])$?
(b) Does $B^0$ have empty interior in $L^1([0, 1])$?
(c) Is $B^1$ a relatively compact subset of $L^1([0, 1])$?

7. Let $f, g : [0, 1] \to \mathbb{R}$ be measurable functions and define the sets $E_k = \{x \in [0, 1] : |f(x)| \leq k\}$ for $k = 1, 2, \ldots$ Assuming that $fg \in L^1([0, 1])$ prove that

$$
\lim_{k \to \infty} \int_{E_k} f(x)g(x)dx = \int_0^1 f(x)g(x)dx.
$$

8. Prove that, for every $\varphi \in C^1_c(\mathbb{R})$,

$$
\lim_{n \to \infty} \int_{\mathbb{R}} \frac{1}{2n} |x|^{\frac{1}{n}-1} \varphi(x)dx = \varphi(0).
$$

9. Set

$$(Tf)(x) = \int_0^1 \frac{f(tx)}{\sqrt{1-t^2}} dt, \quad x \in [0, 1].$$

Prove that $T$ is a linear and continuous operator from the space $C([0, 1])$ into itself and compute its norm.

10. Prove that there exists no function $f : \mathbb{R}^2 \to \mathbb{R}$ of class $C^1$ injective.