The candidate should solve at most five of the following exercises.

(1) Consider the ordinary differential equation in polar coordinates

\[
\begin{cases}
\dot{r} = \begin{cases} 0 & r = 0 \\ \sqrt{r} \sin(1/r) & r \neq 0 \end{cases} \\
\dot{\theta} = f(r),
\end{cases}
\]

with $f \in C^1$ and $0 < \alpha \leq f \leq \beta$.

(a) Study the local and global Lipschitzianity of the right hand side of (1), (2).

(b) For initial data $\theta_0 \in [0, 2\pi]$, $r_0 \in [0, +\infty)$, study the local and global existence and uniqueness of the solution.

(2) Consider the differential equation in $\mathbb{R}$

\[
\dot{x} = x - e^{-t^2}.
\]

Say if there are solutions $x(t)$ such that $x(t) \to 0$ for $t \to \pm\infty$. If the answer is positive, find how many solutions have this property.

(3) Consider the Cauchy problem in $\mathbb{R}$

\[
\dot{x} = x^2(\alpha + \sin(x)), \quad x(0) = 1.
\]

For every $\alpha \in \mathbb{R}$ give an estimate of the maximal interval of definition of the solution.

(4) Consider the linear system of partial differential equations on the torus $\mathbb{T}^2$

\[
\begin{align*}
\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} &= \sin(y) - \cos(x) \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0,
\end{align*}
\]

- Solve the system explicitly, assuming that

\[
\int_0^{2\pi} \int_0^{2\pi} u(x, y) dxdy = \int_0^{2\pi} \int_0^{2\pi} v(x, y) dxdy = 0.
\]
• Prove that if \( f \) is smooth, periodic and with zero average, the solution to the following system on \( \mathbb{T}^2 \)
\[
\begin{align*}
\frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x} &= 0 \\
\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} &= f(x, y)
\end{align*}
\]
satisfies
\[
\int_0^{2\pi} \int_0^{2\pi} \left(u(x, y)u'(x, y) + v(x, y)v'(x, y)\right) dx dy = 0.
\]

(5) In the space \( L^\infty((0, 1), \mathbb{R}) \) consider the set
\[
B = \{ u \in L^\infty((0, 1), \mathbb{R}) : 0 \leq u(x) \leq 1 \text{ almost everywhere} \}.
\]
Let \( E \) be the set made of characteristic functions of unions of open intervals of \( (0, 1) \) with rational end points. Prove that the closure of the convex envelope of \( E \) in the weak* topology coincides with \( B \).

(6) Consider the map \( f : S^1 \to S^1 \) defined as
\[
f(x) = (3x + 2\sin x)/2\pi, \quad x \in S^1.
\]
Prove that for any open (non-empty) set \( A \subseteq S^1 \) there exists \( k \in \mathbb{N} \) such that \( f \circ f \circ \ldots \circ f(A) = S^1 \) \( k \) times.

(7) Let \( S = \{(v_1, v_2, v_3) \in (0, +\infty)^3 : v_1 v_2 v_3 = 1\} \). Given a positive parameter \( \tau \), consider the function \( f_\tau : S \to \mathbb{R} \) defined by
\[
f_\tau(v_1, v_2, v_3) = \left(v_1^2 + v_2^2 + v_3^2 - 3\right) - \tau(v_1 + v_2 + v_3).
\]
Find the number of critical points of \( f_\tau \) for \( \tau \) varying in \( (0, +\infty) \). They represent equilibrium configurations of a cube of incompressible neo-hookean rubber under hydrostatic traction.

(8) Consider the linear operator \( T : L^2([0, 1]) \to L^2([0, 1]) \) given by the formula
\[
(Tf)(x) = \int_0^x f(y) dy \quad \forall f \in L^2([0, 1]).
\]
(a) Prove that the adjoint operator $T^*$ is given by

$$(T^* f)(x) = \int_x^1 f(y) \, dy \quad \forall f \in L^2([0,1]).$$

(b) Prove that

$$(TT^* f)(x) = \int_0^1 \min\{x, y\} f(y) \, dy \quad \forall f \in L^2([0,1]).$$

(c) Compute the spectral radius of $TT^*$ and the norm of $T$.

(9) (a) Let $X$ be a Banach space and let $A \subset X$ be bounded. Prove that $A$ is pre-compact if and only if for every $\epsilon > 0$ there exists a subspace $F_\epsilon$ of $X$ of finite dimension such that

$$\text{dist}(x, F_\epsilon) \leq \epsilon \quad \forall x \in A.$$

(b) Let $(\lambda_n)_{n \in \mathbb{N}}$ be a bounded sequence in $\mathbb{R}$ and consider the linear operator $T : \ell^2 \to \ell^2$ given by

$$(Tu)_n = \lambda_n u_n \quad \forall u = (u_n)_{n \in \mathbb{N}} \in \ell^2.$$ 

Using if necessary point (a) prove that $T$ is compact if and only if

$$\lim_{n \to \infty} \lambda_n = 0.$$

(10) Let $f : \mathbb{R} \to \mathbb{R}$ be a function of class $C^1$ such that $f'(x) \geq 0$ for every $x \in \mathbb{R}$ and $f(0) = 0$. Prove that the solution $x(\cdot)$ of the Cauchy problem

$$\begin{cases}
\dot{x} = \frac{1}{1 + tf(x)} \\
x(0) = 0,
\end{cases}$$

is defined on the whole $\mathbb{R}$ and that

$$\lim_{t \to -\infty} x(t) = -\infty, \quad \lim_{t \to +\infty} x(t) = +\infty.$$