Katz theory on rigid local systems and its extension to KZ equations

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Katz theory on rigid local systems [12], [4], [5] brought a big progress to the theory of Fuchsian ordinary differential equations. The main notions are the rigidity and the middle convolution. A Fuchsian ordinary differential equation is called *rigid* if it is free of accessory parameters. Katz theory provides a simple criterion for the rigidity. We have only to look at the spectral types of the local monodromies. *Middle convolution* is an operation which sends a Fuchsian ODE to another Fuchsian ODE of possibly distinct rank. By using the middle convolution, we can obtain an integral representation of solutions and explicit expressions of connection coefficients for every rigid Fuchsian ODE. Middle convolution can also be applied to non-rigid Fuchsian ODEs, and then we have a new approach to the deformation theory. Oshima [14] developed Katz theory for scalar Fuchsian ODEs, and get many remarkable results. Oshima theory is sometimes more effective than the original Katz theory. In my talk, I will explain: "What is Katz theory" and "How to use Katz-Oshima theory". The references [2] and [7] may be useful.

Secondly, I would like to talk on the extension of Katz theory to the theory on holonomic systems in several variables. The main notions – rigidity and middle convolution can be extended to some class of regular holonomic systems, and we obtain many remarkable results. We also find a new relation between Appell's hypergeometric series F_4 and an algebraic solution to Painlevé VI equation. Recently I succeeded to define the multiplicative middle convolution (i.e. the middle convolution of the monodromy) for Knizhnik-Zamolodchikov (KZ) equations [8]. The monodromy representation of KZ equation is a representation of the pure braid group P_n. Therefore the middle convolution provides a new method of constructing representations of the pure braid group from given ones.

Katz theory is also extended to non-Fuchsian ODEs [1]. We will not mention it. It is a natural and important problem to extend Katz theory to irregular holonomic systems.

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