Entrance Examination 1989-90

1-. A Lie group G acts as a transformation group on the configuration space of a mechanical system. Extending this action to the phase space, show that the (Poisson) algebra of the generating functions of the infinitesimal action is isomorphic to the Lie algebra of G. Give examples of Hamiltonians such that the algebra of constants of the motion is isomorphic to the Lie algebra of some subgroup of the euclidean group $SO(3) \times R^3$.

2-. A particle P of unit mass is gravitationally attracted by a fixed particle Q of mass 2. Use units in which the gravitational constant is one. The initial conditions are: the particle P is at unit distance from Q and its velocity is of unit length and orthogonal to the line PQ. Show that the motion is periodic. Give a numerical estimate of the period, either from above or below, and compare it with the exact result.

3-. Show that, as an operator on $L^2(R^3)$, the domain of each component of the angular momentum J contains the domain of the Hamiltonian H_0 of the harmonic oscillator. Give an explicit computation of the following expressions

$$lim_{T\to\infty} \frac{1}{T}(\psi_a, \int_0^T e^{iJ_3 t} dt \psi_a), \qquad lim_{T\to\infty} \frac{1}{T}(\psi_a, \int_0^T e^{iH_0 t} dt \psi_a)$$

where $\psi_a(x) = e^{-\frac{1}{2}x^2}(a \cdot x + x^2),\ a \in R^3,\ x \in R^3$.

4-. Let Λ be the $2 \times N$ lattice as in the picture below, with interaction energy

$$E(\sigma, \sigma') = -\sum_1^N \sigma_i \sigma'_i - \sum_1^N (\sigma_i \sigma_{i+1} + \sigma'_i \sigma'_{i+1})$$

with $\sigma = \{\sigma_i\},\ \sigma_{N+1} = \sigma_1,\ \sigma'_{N+1} = \sigma'_1,\ \sigma_i = \pm 1$.
Show that the partition function $Z = \sum_{\sigma,\sigma'} exp(-\beta E(\sigma, \sigma'))$ is given by $Z = Tr(A^N)$, where

$$A = \begin{pmatrix} e^{3\beta} & 1 & 1 & e^{-\beta} \\ 1 & e^{\beta} & e^{-3\beta} & 1 \\ 1 & e^{-3\beta} & e^{\beta} & 1 \\ e^{-\beta} & 1 & 1 & e^{3\beta} \end{pmatrix}$$