Entrance Examination 1990-91

1) Given a Lagrangian system with Lagrangian L and a one-parameter group γ_s (with parameter s) of transformations of the configuration space, there is a generalization of the Noether's theorem when

$$\frac{dL}{ds}(q(t),q(t),t) = \frac{df}{dt}(q(t),t)$$

along the trajectories $t \longrightarrow q(t)$ of the system. Using this theorem, find constants of motion for a particle in R^3 if

a - the potential $U(x),\ x \in R^3$ is a homogeneous function of degree 2,

b - the potential reads $U(x,t) = U(x - vt)$, for a constant vector v.

c - the particle moves in a magnetic field with vector potential $A(x)$ which is a homogeneous funtion of degree 1.

Give also a synthetic description (in less than 2 pages) of the most relevant aspects of the Noether's theorem.

2) Given the operator

$$H_\epsilon = (-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2)I + \epsilon cos(x)\sigma_1$$

on $L^2(R) \oplus L^2(R)$ with $I = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ and $\sigma_1 = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$

a - prove that H_ϵ has a pure point spectrum, with finite multiplicities,

b - prove that each eigenfunction of H_ϵ corresponding to a simple eigenvalue has the form $\left(\begin{smallmatrix}\phi_0\\ \phi_0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}\phi_0\\ -\phi_0\end{smallmatrix}\right)$ with $\phi_0 \in L^2(R)$,

c - prove that the lowest eigenvalue for ϵ small (e.g. $0 < \epsilon < 1/10$) is non degenerate

d - for small ϵ, find the expression of the energy of the fundamental state at first order in ϵ.

Give a synthetic account (in less thatn 2 pages) of the theoretical aspects of the problem.

Remark: to answer question b, it may be useful to consider the operator on $L^2(R) \oplus L^2(R)$ given by $\left(\begin{smallmatrix}f\\ g\end{smallmatrix}\right) \rightarrow \left(\begin{smallmatrix}g\\ f\end{smallmatrix}\right)$.

3) In a circular particle accelerator, as a cyclotron or a syncrotron, charged particles move in a magnetic field which keeps them in a circular vacuum chamber. As a first approximation one can assume that the particles' energy is constant, that the magnetic field is axially symmetric, i.e.

$$B(\rho,\phi,z) = B_z(\rho,z)\hat{z} + B_\rho(\rho,z)\hat{\rho}$$

in cylindric coordinates, and that it is purely vertical in the equatorial plane $B(\rho,\phi,0) = 0$. Describe the structure of the orbits in the equatorial plane and discuss their linear stability.