1) Given a Lagrangian system with Lagrangian $L$ and a one-parameter group $\gamma_s$ (with parameter $s$) of transformations of the configuration space, there is a generalization of the Noether’s theorem when

$$\frac{dL}{ds}(q(t), q(t), t) = \frac{df}{dt}(q(t), t)$$

along the trajectories $t \rightarrow q(t)$ of the system. Using this theorem, find constants of motion for a particle in $\mathbb{R}^3$

- the potential $U(x)$, $x \in \mathbb{R}^3$ is a homogeneous function of degree 2,
- the potential reads $U(x, t) = U(x - vt)$, for a constant vector $v$.
- the particle moves in a magnetic field with vector potential $A(x)$ which is a homogeneous function of degree 1.

Give also a synthetic description (in less than 2 pages) of the most relevant aspects of the Noether’s theorem.

2) Given the operator

$$H_\epsilon = \left(-\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2\right) I + \epsilon \cos(x) \sigma_1$$

on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- prove that $H_\epsilon$ has a pure point spectrum, with finite multiplicities,
- prove that each eigenfunction of $H_\epsilon$ corresponding to a simple eigenvalue has the form $\begin{pmatrix} \phi_0 \\ \phi_0 \end{pmatrix}$ or $\begin{pmatrix} -\phi_0 \\ \phi_0 \end{pmatrix}$ with $\phi_0 \in L^2(\mathbb{R})$,
- prove that the lowest eigenvalue for $\epsilon$ small (e.g. $0 < \epsilon < 1/10$) is non degenerate
- for small $\epsilon$, find the expression of the energy of the fundamental state at first order in $\epsilon$.

Give a synthetic account (in less than 2 pages) of the theoretical aspects of the problem.

Remark: to answer question b, it may be useful to consider the operator on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ given by $(f, g) \rightarrow (f, g)$

3) In a circular particle accelerator, as a cyclotron or a synchrotron, charged particles move in a magnetic field which keeps them in a circular vacuum chamber. As a first approximation one can assume that the particles’ energy is constant, that the magnetic field is axially symmetric, i.e.

$$B(\rho, \phi, z) = B_z(\rho, z) \hat{z} + B_\phi(\rho, z) \hat{\phi}$$

in cylindric coordinates, and that it is purely vertical in the equatorial plane $B(\rho, \phi, 0) = 0$. Describe the structure of the orbits in the equatorial plane and discuss their linear stability.