Mathematical Physics Sector
Entrance Examination 1995/96

1) Let us consider a holonomic mechanical system, subject to time-independent constraints and conservative forces. The kinetic energy defines a Riemannian metric T_2 on the configuration space Q of the system.

a) Describe the Legendre transform as an application

$$\mathcal{L} : TQ \to T^*Q$$

where TQ and T^*Q are the tangent and cotangent bundle to Q, respectively. Compare $\mathcal{L}$ with the isomorphism $TQ \to T^*Q$ induced by T_2.

b) Under the assumption that there are no active forces, show the the geodesics of the Riemannian manifold (Q, T_2) coincide with the curves which describe the time evolution of the system.

c) Assume now that the mechanical system is a rigid body with a fixed point (so $Q \simeq SO(3)$). Write the Euler equations of motion of the rigid body in terms of the Lie bracket of the Lie algebra of the group $SO(3)$.

2) Define a Laplace oparator on the interval $(0, 1)$ and study its perturbation given by the potential

$$\begin{aligned} V^N(x) =& 1 \; for \; \frac{n}{N} \leq x < \frac{n+\frac{1}{2}}{N}, \; (n = 0, ..., N-1) \\ =& 0 \; otherwise. \end{aligned}$$

In particular, discuss the convergence of the perturbative series for the resolvent and compute explicitely some terms for $N = 1$.

Discuss the limit $N \to \infty$, find the limit operator and comment on the type of convergence.

3) Consider piecewise linear random walks in dimension d. Assume that the transition function for moving on a straight trajectory from the point x to the point y is given by

$$t_\beta(x \to y) = \exp\left(-\beta|x-y|\right)$$

Define the regularized propagator as

$$G_\beta(x, y) = \sum_{N=0}^{\infty} \int dx_1 \ldots dx_N t_\beta(x \to x_1) \ldots t_\beta(x_N \to y)$$

If c_0 and σ_0 denote the 0^{th} and the 2^{th} moment of t_β provide an approximate expression for $G_\beta(x, y)$, discuss its convergence properties for different values of β. Study the continuum limit at the critical point.

4) Describe the behavior of uniformly accelerated observers in Minkowski spacetime. Discuss the global structure of the spacetime manifold as seen by these observers. Illustrate with examples the notions of event horizon and apparent horizon.