• We consider the spaces $\mathbf{R}^n$, $n \geq 1$ with their standard topology.

(i) Prove by elementary reasoning that if $f : \mathbf{R} \to \mathbf{R}^2$ is a one-to-one continuous map then $f(\mathbf{R})$ cannot be open in $\mathbf{R}^2$.

(ii) Show an example of a one-to-one continuous map $f : \mathbf{R} \to \mathbf{R}^2$ such $f(\mathbf{R})$ with the relative topology is not homeomorphic to $\mathbf{R}$.

(iii) More generally, show that if a homeomorphism $f : \mathbf{R}^m \to \mathbf{R}^n$ exists, then $m = n$.

• After briefly expounding the general theory of the Hamilton-Jacobi equation, the candidate should solve the following exercise.

The configuration space of a mechanical system is diffeomorphic to $\mathbf{R}^n$. One knows that the function

$$S(q^1, \dots, q^n, \alpha_1, \dots, \alpha_n, t) = e^{-\beta t} \sum_{i=1}^{n} q^i \alpha_i$$

solves the Hamilton-Jacobi equation ($\beta$ is a real costant, and the quantities $\alpha_1, \dots, \alpha_n$ are integration constants).

(i) Write the Hamiltonian function as a function of $(q, p)$.

(ii) Solve the Hamilton-Jacobi equation associated to this Hamiltonian function by additive separation of variables; compare with $S$.

(iii) Find a solution of the Hamilton-Jacobi equation where the variables $q_1, \dots, q_n$ are not separated.

(iv) Check that the solutions of the Hamilton-Jacobi equation so obtained are constant along the solutions of the equations of motion; give theoretical reasons why if a solution is constant also the others must be.

• A quantum free particle moves freely on the real axis between two walls situated at the points $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, with bouncing (Neumann) boundary conditions. Use units of measure where $m = h/2\pi = 1$.

At $t = 0$ the configuration is described by the wave function

$$\phi_0(x) = \sqrt{\frac{1 - \epsilon^2}{\pi}} + \epsilon \sqrt{\frac{2}{\pi}} \sin\, x$$

At the time $t_1 = \pi$ the walls are suddenly removed and then the particle moves freely on the whole real axis. At the time $t_2 = t_1 + 1$ a position measurement is done.

Compute the probability that the numerical result of the measurement is included between $-\pi/2$ and $\pi/2$.

• In Minkowski space-time $M^4$ let us consider a uniformly rotating frame of reference $R_\omega$. Describe the space-time geometry in $R_\omega$ and characterize the metric $g^2_\omega$ in the two-dimensional plane $\pi_\omega$, comoving with $R_\omega$, and orthogonal to the rotation axis; prove that the geometry $(\pi_\omega, g^2_\omega)$ so obtained is isometric to a surface of constant negative curvature.