Entrance Examination 1997/98

The candidate is expected to solve one of the following three problems:

Exercise 1)

Consider the phase space $\mathbf{R}^4$ with canonical coordinates $\{x_1, x_2, p_1, p_2\}$ ($\{x_i, p_j\} = \delta_{ij}$, all other Poisson brackets vanishing), and the Hamiltonian function locally given by

$$H_f := \frac{1}{2}(p_1^2 + p_2^2) + \frac{k^2}{(x_1 - x_2)^2} + f(x_1, x_2).$$

1a) For $f(x_1, x_2) = (x_1^2 + x_2^2)/2$, find the critical points of the hamiltonian system induced by H_f, determine their type and find the frequency of the small oscillations.

1b) Let $p = p_1 - p_2$, $q = x_1 - x_2$, $P = p_1 + p_2 = 0$ and consider an infinitesimal $f(p, q) = \epsilon\phi(q, p)$. Find the equations for the generating function $\sigma(p, q)$ of an *infinitesimal* canonical transformation $\{p, q\} \to \{r, s\}$ such that (up to the first order in ϵ) the new hamiltonian has the form

$$\tilde{H}(r, s) = \frac{1}{4}r^2 + \frac{k^2}{s^2}.$$

1c) Consider the unperturbed Hamiltonian H_0. Let L be the map from $\mathbf{R}^4$ to the space of 2 by 2 matrices given by:

$$L(x_1, x_2, p_1, p_2) := \begin{bmatrix} p_1 & \dfrac{k}{(x_1 - x_2)} \\ \dfrac{k}{(x_1 - x_2)} & p_2 \end{bmatrix}$$

Find another matrix B (hint: set $B_{11} = B_{22} = 0$) such that, along the Hamitonian flows induced by H_0, L evolves as

$$\frac{d}{dt}L = [L, B]$$

($[\cdot, \cdot]$ is the matrix commutator).

Discuss to what extent the resulting equations of motion are equivalent to the original ones.

Show that the eigenvalues (λ_1, λ_2) of L are mutually commuting constants of the motion. Hint: Remind that, for any 2 by 2 matrix M:

$$det(M - \mu\mathbf{I}) = \mu^2 - \mu \cdot Tr(M) + \frac{1}{2}(Tr(M^2) - (Tr(M))^2)$$

Exercise 2)

Suppose that M is the configuration space of a mechanical system with a finite number of degrees of freedom, and let G be a Lie group acting on M by diffeomorphisms. Let $L : TM \to \mathbf{R}$ be a function invariant under the induced action of G on TM. (Here TM is the tangent bundle to M.)

2.a) State the Noether theorem which holds in this situation.

2.b) Work out in detail the case where M is the configuration space of a rigid body with a fixed point, and $L = K - V$, where K is the kinetic energy, and the potential energy V depends only on the height of the centre of mass.

2.c) Give a generalization of Noether theorem to classical field theory (again considering the action of a finite-dimensional Lie group), and sketch a geometric setting for it.

Exercise 3)

Consider the operators $A = -i\partial/\partial x$ and $B = -i\partial/\partial y$ defined on the domain $\mathcal{D} = C_o^\infty(\Sigma - \{0\})$ (smooth functions with support not including 0) in the Hilbert space $\mathcal{H} = L^2(\Sigma\ ,\ dxdy\ )$, where Σ is the Riemann surface of $\sqrt{z}$ and $z = x + iy$. Show that

3a) the domain D is preserved by A and B.

3b) A and B commute on D

3c) A and B are essentially self-adjoint

3d) $\exp(i\bar{A}t)$ and $\exp(i\bar{B}t)$ do not commute in $\mathcal{H}$ for some $t \in \mathbf{R}$, where $\bar{A}$ and $\bar{B}$ denote the closure of, respectively, A and B.