Mathematical Physics Sector
Entrance examination 1998/1999

The candidate is asked to solve at least one of the following problems.

Problem No. 1

1) Give a brief outline of the theory of the Hamilton–Jacobi equation.

2) Consider a Hamiltonian of the form

$$h = \frac{\sum_{k=1}^{n} f_k(q_k, p_k)}{\sum_{k=1}^{n} g_k(q_k, p_k)}$$

where $f_k,\ g_k$ are smooth functions of two variables for $k = 1, \ldots, n$.

2a) Show that the associated Hamilton–Jacobi equation can be solved by the method of separation of variables.

2b) Prove that the functions $F_0 = h;\ F_k = f_k - h \cdot g_k,\ k = 1, \ldots, n-1$ are involutive conserved quantities of the Hamiltonian system:

$$\dot{p}_i = -\frac{\partial h}{\partial q_i}$$
$$\dot{q}_i = \frac{\partial h}{\partial p_i}$$

3) Let

$$H = p_x^2 + \frac{p_y^2}{x^2} + \frac{y}{x^2}.$$

Use the solution of the Hamilton–Jacobi equation associated to H to find the solutions $x(t)$, $y(t)$ of the equation of motions. (Recall that integrals of the form $\int F(\sqrt{ax^2 + bx + c}; x)dx$ can be computed by means of the Euler substitution $t = \sqrt{ax^2 + bx + c} \pm \sqrt{a} \cdot x$).

Problem No. 2

Let $f(x)$ be a continuous doubly periodic function on the plane, i.e., there are two linearly independent vectors a, b ("elementary periods") such that

$$f(x + ma + nb) = f(x)\ \forall x,$$

for arbitary integers m, n.

1) Prove that, if a non–constant $f(x)$ satisfies also $f(x + mc) = f(x)$ for all x and for any integer m, then the vector c must be of the form $c = pa + qb$ with rational p, q.

2) Suppose now that a continuos doubly periodic function $f(x)$ satisfies $f(R(x)) = f(x)$ where R is the rotation of the plane around some point by the angle $\pi/5$. Prove that $f(x)$ is a constant.

Problem No. 3

1) Briefly describe the properties of the group of Lorentz transformations of Minkowski space–time, i.e the group of linear transformations Λ

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

such that $\Lambda^T \eta \Lambda = \eta$, η being the matrix $diag(1, -1, -1, -1)$.

2) Knowing that under an arbitrary Lorentz transformation Λ the vectors $\mathbf{E}$ and $\mathbf{B}$ (electric and magnetic fields) transform into vectors $(\mathbf{E}^\Lambda, \mathbf{B}^\Lambda)$ in such a way that

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y, & B_x & 0 \end{bmatrix}$$

behaves as rank two contravariant tensor, i.e. $F \mapsto F^\Lambda = \Lambda F \Lambda^T$:

2a) Prove that $|\mathbf{E}|^2 - |\mathbf{B}|^2$ and $|\mathbf{E} \cdot \mathbf{B}|$ are Lorentz invariant and that the duality transformation $(\mathbf{E}, \mathbf{B}) \mapsto (-\mathbf{B}, \mathbf{E})$ commutes with the proper orthocronous Lorentz group $L_+^\uparrow = \{\Lambda \ s.t., \Lambda_0^0 \geq 1, \ det\Lambda = 1\})$.

2b) Show that for every $\Lambda \in L_+^\uparrow$ there exists a 3×3 complex orthogonal matrix $A(\Lambda)$ such that

$$\mathbf{E}^\Lambda + i\mathbf{B}^\Lambda = A(\Lambda) \cdot (\mathbf{E} + i\mathbf{B})$$

2c) Write the transformation law for $\mathbf{E}$ and $\mathbf{B}$ under a Lorentz boost

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} + \left[(\gamma - 1)\frac{\mathbf{x} \cdot \mathbf{v}}{v^2} - \gamma t\right] \mathbf{v} \\ t' &= \gamma \left(t - \frac{\mathbf{x} \cdot \mathbf{v}}{c^2}\right) \end{aligned} .$$

2d) Consider the case in which $\mathbf{E}$ and $\mathbf{B}$ are uniform and mutually orthogonal. Discuss under which conditions there exists a Lorentz boost Λ for which $\mathbf{E}^\Lambda = 0$ or $\mathbf{B}^\Lambda = 0$.

Problem No. 4

Consider a system of linear differential equations

$$\frac{dy}{dz} = \left(\sum_{i=1}^{k} \frac{A_i}{z - z_i} \right) y \qquad (*)$$

for the vector-function

$$y(z) = \begin{pmatrix} y_1(z) \\ y_2(z) \\ \cdot \\ \cdot \\ \cdot \\ y_n(z) \end{pmatrix}.$$

Here $A_1, \ldots, A_k$ are z-independent $n \times n$ matrices, $z_1, \ldots, z_k$ are pairwise distinct complex numbers. Let

$$Y(z) = \begin{pmatrix} y_{11}(z) & \cdot & \cdot & \cdot & y_{1n}(z) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ y_{n1}(z) & \cdot & \cdot & \cdot & y_{nn}(z) \end{pmatrix}$$

be a fundamental matrix of the system (*) (i.e., the columns of $Y(z)$ form a basis in the space of solutions of (*)) defined in a neighborhood of a point $z = z_0$, $z_0 \neq z_i$ for any $i = 1, \ldots, k$.

1) Prove that $\det Y(z)$ does not depend on z if $\operatorname{trace}(A_i) = 0$, $i = 1, \ldots, k$.

2) Let γ be an oriented closed loop on the z-plane passing through z_0 but avoiding the points $z_1, \ldots, z_k$. Denote $Y_\gamma(z)$ the result of analytic continuation of the fundamental matrix $Y(z)$ along γ. Prove that there exists a nondegenerate $n \times n$ matrix M_γ such that

$$Y_\gamma(z) = Y(z) M_\gamma.$$

3) Consider the particular case $n = 2$. Let us assume that

— $\operatorname{trace}(A_i) = 0$, $\det A_i = 0$, $i = 1, \ldots, k$.

— for any two closed loops γ_1, γ_2 the matrices M_{γ_1} and M_{γ_2} commute:

$$M_{\gamma_1} M_{\gamma_2} = M_{\gamma_2} M_{\gamma_1}.$$

Prove that there exists a nondegenerate 2×2-matrix T such that the matrices

$$B_i := T^{-1} A_i T$$

have the form

$$B_i = \begin{pmatrix} 0 & b_i \\ 0 & 0 \end{pmatrix}$$

for any $i = 1, \ldots, k$.