The candidate is asked to solve at least one of the following problems.

**Problem No. 1**

1) Give a brief outline of the theory of the Hamilton–Jacobi equation.

2) Consider a Hamiltonian of the form

\[ h = \sum_{k=1}^{n} f_k(q_k, p_k) \]

where \( f_k, g_k \) are smooth functions of two variables for \( k = 1, \ldots, n \).

2a) Show that the associated Hamilton–Jacobi equation can be solved by the method of separation of variables.

2b) Prove that the functions \( F_0 = h; F_k = f_k - h \cdot g_k, k = 1, \ldots, n - 1 \) are involutive conserved quantities of the Hamiltonian system:

\[ \dot{q}_i = \frac{\partial h}{\partial p_i} \]

\[ \dot{p}_i = -\frac{\partial h}{\partial q_i} \]

3) Let

\[ H = \frac{p_x^2}{x^2} + \frac{p_y^2}{y^2} + \frac{y}{x^2}. \]

Use the solution of the Hamilton–Jacobi equation associated to \( H \) to find the solutions \( x(t), y(t) \) of the equation of motions. (Recall that integrals of the form \( \int F(\sqrt{ax^2 + bx + c}; x)dx \) can be computed by means of the Euler substitution \( t = \sqrt{ax^2 + bx + c} \pm \sqrt{a} \cdot x \).)

**Problem No. 2**

Let \( f(x) \) be a continuous doubly periodic function on the plane, i.e., there are two linearly independent vectors \( a, b \) ("elementary periods") such that

\[ f(x + ma + nb) = f(x) \quad \forall x, \]

for arbitrary integers \( m, n \).
1) Prove that, if a non-constant \( f(x) \) satisfies also \( f(x + mc) = f(x) \) for all \( x \) and for any integer \( m \), then the vector \( c \) must be of the form \( c = pa + qb \) with rational \( p, q \).

2) Suppose now that a continuous doubly periodic function \( f(x) \) satisfies \( f(R(x)) = f(x) \) where \( R \) is the rotation of the plane around some point by the angle \( \pi / 5 \). Prove that \( f(x) \) is a constant.

**Problem No. 3**

1) Briefly describe the properties of the group of Lorentz transformations of Minkowski space–time, i.e. the group of linear transformations \( \Lambda \)

\[
\begin{pmatrix}
  c t' \\
  x' \\
  y' \\
  z'
\end{pmatrix}
= \Lambda
\begin{pmatrix}
  c t \\
  x \\
  y \\
  z
\end{pmatrix}
\]

such that \( \Lambda^T \eta \Lambda = \eta \), \( \eta \) being the matrix \( \text{diag}(1, -1, -1, -1) \).

2) Knowing that under an arbitrary Lorentz transformation \( \Lambda \) the vectors \( E \) and \( B \) (electric and magnetic fields) transform into vectors \( (E^\Lambda, B^\Lambda) \) in such a way that

\[
F^{\mu\nu} = \begin{bmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & -B_z & B_y \\
E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0
\end{bmatrix}
\]

behaves as rank two contravariant tensor, i.e. \( F \mapsto F^\Lambda = \Lambda F \Lambda^T \):

2a) Prove that \( |E|^2 - |B|^2 \) and \( E \cdot B \) are Lorentz invariant and that the duality transformation \( (E, B) \mapsto (-B, E) \) commutes with the proper orthochronous Lorentz group \( L_+^\Lambda = \{ \Lambda \text{ s.t. } A_0 > 1, \ det \Lambda = 1 \} \).

2b) Show that for every \( \Lambda \in L_+^\Lambda \) there exists a \( 3 \times 3 \) complex orthogonal matrix \( A(\Lambda) \) such that

\[
E^\Lambda + iB^\Lambda = A(\Lambda) \cdot (E + iB)
\]

2c) Write the transformation law for \( E \) and \( B \) under a Lorentz boost

\[
\begin{align*}
x' &= x + \left[(\gamma - 1)\frac{x \cdot v}{v^2} - \gamma t\right]v \\
t' &= \gamma \left(t - \frac{x \cdot v}{c^2}\right)
\end{align*}
\]

2d) Consider the case in which \( E \) and \( B \) are uniform and mutually orthogonal. Discuss under which conditions there exists a Lorentz boost \( \Lambda \) for which \( E^\Lambda = 0 \) or \( B^\Lambda = 0 \).

**Problem No. 4**
Consider a system of linear differential equations

\[ \frac{dy}{dz} = \left( \sum_{i=1}^{k} \frac{A_i}{z - z_i} \right) y \tag{*} \]

for the vector-function

\[ y(z) = \begin{pmatrix} y_1(z) \\ y_2(z) \\ \vdots \\ y_n(z) \end{pmatrix} \]

Here \( A_1, \ldots, A_k \) are \( z \)-independent \( n \times n \) matrices, \( z_1, \ldots, z_k \) are pairwise distinct complex numbers. Let

\[ Y(z) = \begin{pmatrix} y_{11}(z) & \cdots & y_{1n}(z) \\ \vdots & \ddots & \vdots \\ y_{n1}(z) & \cdots & y_{nn}(z) \end{pmatrix} \]

be a fundamental matrix of the system \((*)\) (i.e., the columns of \( Y(z) \) form a basis in the space of solutions of \((*)\)) defined in a neighborhood of a point \( z = z_0, z_0 \neq z_i \) for any \( i = 1, \ldots, k \).

1) Prove that \( \det Y(z) \) does not depend on \( z \) if \( \text{trace}(A_i) = 0, i = 1, \ldots, k \).

2) Let \( \gamma \) be an oriented closed loop on the \( z \)-plane passing through \( z_0 \) but avoiding the points \( z_1, \ldots, z_k \). Denote \( Y_\gamma(z) \) the result of analytic continuation of the fundamental matrix \( Y(z) \) along \( \gamma \). Prove that there exists a nondegenerate \( n \times n \) matrix \( M_\gamma \) such that

\[ Y_\gamma(z) = Y(z)M_\gamma. \]

3) Consider the particular case \( n = 2 \). Let us assume that

- \( \text{trace}(A_i) = 0, \det A_i = 0, i = 1, \ldots, k. \)
- for any two closed loops \( \gamma_1, \gamma_2 \) the matrices \( M_{\gamma_1} \) and \( M_{\gamma_2} \) commute:

\[ M_{\gamma_1}M_{\gamma_2} = M_{\gamma_2}M_{\gamma_1}. \]

Prove that there exists a nondegenerate \( 2 \times 2 \) matrix \( T \) such that the matrices

\[ B_i := T^{-1}A_iT \]

have the form

\[ B_i = \begin{pmatrix} 0 & b_i \\ 0 & 0 \end{pmatrix} \]

for any \( i = 1, \ldots, k \).