SISSA ISAS

SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

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Entrance examination 1999/2000

The candidate is asked to solve one of the following exercises.

1 Theory of matrices

A. Briefly recall the basic properties of eigenvalues and eigenvectors of real symmetric matrices.
B. Let

$$a_n, \ b_n, \quad -\infty < n < \infty$$

be two periodic sequences of real numbers, i.e.

$$a_{n+N} = a_n, \ b_{n+N} = b_n \ \text{ for any } n.$$

The positive integer N is called period of the sequences. We assume all numbers a_n to be nonzero.

Consider a system of linear equations depending on a parameter λ

$$a_n\psi_{n-1} + b_n\psi_n + a_{n+1}\psi_{n+1} = \lambda\psi_n, \quad -\infty < n < \infty \tag{1.1}$$

in the space of complex valued sequences $(\psi_n)_{-\infty<n<\infty}$.

1). Prove that for any complex λ there exist a nonzero solution (ψ_n) to 1.1 satisfying

$$\psi_{n+N} = \mu\,\psi_n \ \text{ for any } n \tag{1.2}$$

with some complex coefficient μ.

2). Prove that for any complex λ except for at most $2N$ values $\lambda_1, \ldots, \lambda_{2m}$, $m \le N$, there exist exactly 2 nonzero solutions to the system (3.1, 3.2) with two different values of μ.

3). Prove that all the exceptional values $\lambda_1, \ldots, \lambda_{2m}$ are real.

2 Classical field theory

A possibile Lagrangian density for the electromagnetic field in empty space is

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}\, F^{\mu\nu}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor and A_μ is the electromagnetic potential. One should recall that the expression of $F_{\mu\nu}$ in terms of the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ is

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

Greek indices run from 0 to 3, Latin ones from 1 to 3, and the spacetime metric signature is $(+---)$. The C.G.S. system of units is used. Whenever necessary one should assume that the electromagnetic field is different from zero only in a bounded region of three-space.

1) Write the canonical stress-energy tensor $t^{\mu\nu}$ associated with the Lagrangian $\mathcal{L}$.

2) Prove that the tensor

$$T^{\mu\nu} = t^{\mu\nu} - \frac{1}{4\pi} F^{\alpha\mu}\, \partial_\alpha A^\nu$$

is symmetric and yields the same conservation laws as $t^{\mu\nu}$.

3) Write the components of $T^{\mu\nu}$ in terms of the fields $\mathbf{E}$ and $\mathbf{B}$.

4) Discuss the conservation laws associated with the tensor $T^{\mu\nu}$, comparing with Poynting's theorem.

5) After defining the angular momentum tensor

$$M^{\alpha\beta\gamma} = T^{\alpha\beta}\, x^\gamma - T^{\alpha\gamma}\, x^\beta$$

prove that the equations $\partial_\alpha\, M^{\alpha ij} = 0$ are equivalent to the conservation of the angular momentum of the electromagnetic field. The latter is defined as the vector

$$\mathbf{L} = \frac{c}{4\pi} \int_{x^0=K} \mathbf{x} \wedge (\mathbf{E} \wedge \mathbf{B})\, d^3x\,.$$

where K is a constant.

6) Prove that the conservation laws

$$\partial_\alpha M^{\alpha 0 i} = 0$$

are equivalent to

$$\frac{\mathcal{E}}{c^2}\frac{d\mathbf{X}}{dt} = \mathbf{P}$$

where

$$\mathcal{E} = \int_{x^0=K} T^{00}\, d^3x, \qquad X^i = \frac{1}{\mathcal{E}}\int_{x^0=K} x^i\, T^{00} d^3x, \qquad P^i = \int_{x^0=K} T^{0i}\, d^3x\,.$$

3 Topology

A. Formulate and discuss the Poincaré lemma for differential forms defined in domains of Euclidean space.

B.

1. Let $(v^1(x), \ldots, v^n(x))$ be a smooth vector field defined for x in a ball B of n-dimensional Euclidean space. Prove that, if

$$\sum_{i=1}^{n} \frac{\partial v^i(x)}{\partial x^i} = 0 \tag{3.1}$$

then the components of the vector field can be represented in the form

$$v^i(x) = \sum_{j=1}^{n} \frac{\partial \omega^{ij}(x)}{\partial x^j} \tag{3.2}$$

for an antisymmetric smooth tensor field $\omega^{ij}(x) = -\omega^{ji}(x)$ defined in the ball B.

2. Suppose now that the vector field $(v^1(x), \ldots, v^n(x))$ satisfy Eq. (3.1) in a domain B of Euclidean space that is not a ball. Can one always represent it in the form (3.2) for some tensor field $\omega^{ij}(x)$? Discuss the topological properties of B that could obstruct such a reducibility.

3. Construct an example of a domain B of some dimension $n \geq 2$ and of a vector field satisfying (3.1) in B which cannot be represented in the form (3.2).

4 Analytical Mechanics

A) Briefly outline the theory of generating functions for symplectic (that is, canonical and independent of time) transformations in Classical Mechanics.
Consider in $\mathbb{R}^2 = T^*\mathbb{R}$, endowed with canonical coordinates (p, q), the vector field

$$X = \begin{cases} \dot{q} = p - q^2 \\ \dot{p} = 2qp - 2q^3 - \text{Sinh}(q) \end{cases}$$

1) Prove that X is a hamiltonian vector field.
2) Find a symplectic transformation under which the Hamiltonian of X takes the form of the energy of a particle of unit mass on the line $\mathbb{R}$ in a suitable potential.
3) Find a generating function for such a transformation.

B) Briefly recall the formal properties of the Poisson brackets (i.e. antisymmetry, Leibnitz rule, ...) seen as a bilinear composition law for functions on an arbitrary manifold. Express them in local coordinates and show that the standard rule for canonical coordinates $(p_i, q_i), i = 1, \ldots, n$

$$\{p_j, q_k\} = \delta_{jk}, \ \{p_j, p_k\} = 0, \ , \{q_j, q_k\} = 0$$

fullfills those properties.
Consider now cylindical coordinates (ρ, θ, z) on a suitable open set $U \subset \mathbb{R}^3$.
1) Verify that the assignement:

$$\{\rho, \theta\}_1 = \frac{z}{\rho}, \ \{\theta, z\}_1 = 1, \ \{\rho, z\}_1 = 0 \tag{4.1}$$

satisfies the properties of a Poisson bracket.
2) Find a (class of) function(s) $F(\rho, \theta, z)$ such that

$$\{F, G\}_1 = 0 \, \forall \, G \in C^\infty(U)$$

3) Is there any vector field Y on U, representable in the form

$$Y := \left\{ \dot{\rho} = \{K, \rho\}_1 \quad \dot{\theta} = \{K, \theta\}_1 \quad \dot{z} = \{K, z\}_1 \right\}$$

where K is a smooth function on U (that is, Y is a *Hamiltonian vector field* with respect to the brackets $\{\cdot, \cdot\}_1$), whose flow connects the points $u_0 = (1, \pi/2, -1)$ and $u_1 = (1/2, 2\pi/3, 2)$?