Mathematical Physics Sector
Entrance examination 2002/2003 – October Session

The candidate is asked to solve at least one problem among the following.

1 Quantum Mechanics

A Harmonic oscillator in an electric field
Consider the one-dimensional quantum harmonic oscillator with Hamiltonian

\[ H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \]

The particle has also a charge \( e \). Switching on a constant electric field \( \mathcal{E} \), the Hamiltonian gets an additional potential term and becomes

\[ H'_- = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - e\mathcal{E}x \tag{1.1} \]

1. Using the well-known expression of the energy spectrum of the harmonic oscillator, determine the spectrum of the new Hamiltonian.

2. Prove that the two operators \( H'_- \) and \( H'_+ \) (obtained by changing \( \mathcal{E} \rightarrow -\mathcal{E} \) in eq.(1.1) are unitarily equivalent in \( \mathcal{L}_2(-\infty, \infty) \).

3. Determine the value of \( \mathcal{E} \) such that the energy of the first excited state \( |\psi'_1\rangle \) of \( H'_- \) coincides with the energy of ground state \( |\psi_0\rangle \) of \( H \). Which observable permits to distinguish \( |\psi'_1\rangle \) from \( |\psi_0\rangle \)?
2 Linear algebra

Linear operators and matrices

Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear operator acting in the $n$-dimensional real space $\mathbb{R}^n$. Denote

$$A = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}$$

the matrix of the linear operator with respect to some basis $e_1, e_2, \ldots, e_n$ in the space $\mathbb{R}^n$, i.e.

$$A e_j = \sum_{i=1}^{n} a_{ij} e_i, \quad j = 1, 2, \ldots, n.$$  

Denote $P_A(t)$ the characteristic polynomial of the operator $A$,

$$P_A(t) = \det(A - tI).$$

Here

$$I = \begin{pmatrix}
    1 & 0 & \cdots & 0 \\
    0 & 1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 1
\end{pmatrix}$$

is the identity matrix.

1. Considering the complex numbers $\mathbb{C}$ as the two-dimensional real space $\mathbb{R}^2$, introduce the linear operator of multiplication by a given complex number $\lambda$,

$$A z = \lambda z, \quad z \in \mathbb{C}.$$  

Compute the characteristic polynomial $P_A(t)$.

2. Let

$$A, B : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

be two linear operators acting in the two-dimensional real space satisfying

$$A B = B A.$$  

Prove the existence of such a basis $e_1, e_2$ that the matrices $A$, $B$ of the linear operators take one of the following forms:
\[
A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \quad (2.1)
\]

or
\[
A = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{pmatrix} \quad (2.2)
\]

or
\[
A = \begin{pmatrix} a_1 & 1 \\ 0 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix} \quad (2.3)
\]

for some real numbers \(a_1, a_2, b_1, b_2\).

3. Let two linear operators
\[
\mathcal{A}, \mathcal{B} : \mathbb{R}^3 \to \mathbb{R}^3
\]
acting in three-dimensional real space satisfy the following equation
\[
\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} = \mathcal{B}.
\]

Compute the characteristic polynomial \(P_B(t)\).
3 Classical mechanics

A Classical Particle in a magnetic field
Consider the system defined in the Euclidean space \( \mathbb{R}^n \) by the Lagrangian

\[
L_A = \frac{m}{2} \dot{x} \cdot \dot{x} + \frac{q}{c} \dot{x} \cdot A(x) - V(x),
\]

where \( x = (x_1, \ldots, x_n) \) are Cartesian coordinates, \( A(x) \) is the vector potential, \( V(x) \) is the scalar potential, and \( \dot{x} \) denotes the velocity vector.

1) Prove that the Euler Lagrange equations associated with \( L_A \) are invariant under the gauge transformation

\[
A(x) \rightarrow A'(x) = A(x) + \text{grad} \, \Phi(x)
\]

2) Prove that, in the Hamiltonian formalism, the transformation (3.2) yields a canonical transformation.

3) Consider the case of \( \mathbb{R}^3 \) endowed with Cartesian coordinates \( x, y, z \). Use the above arguments to prove complete integrability (that is, to find an integral of the motion in addition to the obvious two) of the system with Lagrangian

\[
L_B = \frac{m}{2} (x^2 + y^2 + z^2) - \frac{q}{c} By \dot{x} - V(\rho),
\]

where \( B \) is a constant and \( \rho = \sqrt{x^2 + y^2} \).

(Hint: compute first the magnetic field associated with the vector potential \( A = [-By, 0, 0] \).

4) Consider the Hamiltonian of the system (3.3), written in the appropriate gauge, and discuss qualitatively the resulting one dimensional radial motion, assuming \( V(\rho) = \log \rho \), for generic values of the constants of the motion.
4 Statistical Mechanics

One-dimensional random walk
At discrete units of time \((t = 1, 2, 3, \ldots)\), a particle moves along a one-dimensional lattice. Suppose it starts at the origin and that at each step it moves either one lattice site to the right or to the left, each with probability \(1/2\).

For \(n \geq 1\), let \(u_n\) be the probability that the particle returns to the origin at time \(t = n\) and \(p_n\) the probability that the first return to the origin occurs at \(t = n\). For convenience we take \(p_0 = 0\) and \(u_0 = 1\). Clearly \(p_n = u_n = 0\) whenever \(n\) is odd.

1. Show that for \(n \geq 1\), the two probabilities are related as

\[
u_n = p_0 u_n + p_1 u_{n-1} + p_2 u_{n-2} + \cdots + p_n u_0 \tag{4.1}\]

2. Prove that

\[
u_{2n} = \frac{1}{2^{2n}} \binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \tag{4.2}\]

3. Show that the particle will sooner or later return to its initial position, i.e.

\[p_2 + p_4 + p_6 + \cdots = 1\]

Hint. Introduce the generating functions

\[F(x) = \sum_{n=0}^{\infty} p_n x^n, \quad U(x) = \sum_{n=0}^{\infty} u_n x^n\]

and show that eq. (4.1) leads to the relation \(U(x) = 1 + F(x)U(x)\). Use this relation and the \(u_{2n}\) given by eq. (4.2) to determine \(F(x)\). It is useful to remind that

\[(1 + x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \cdots + \frac{q(q-1)\cdots(q-k+1)}{k!}x^k + \cdots\]
5 Geometry

Projective spaces
Let $\mathbb{P}^n(\mathbb{C})$ be the $n$-dimensional complex projective space ($n \geq 1$).

1. Prove that $\mathbb{P}^1(\mathbb{C})$ is homeomorphic to the real 2-dimensional sphere $S^2$.

2. Show that $\mathbb{P}^n(\mathbb{C})$ contains a copy of $\mathbb{C}^n$ in such a way that the complement is isomorphic to $\mathbb{P}^{n-1}(\mathbb{C})$.

3. Compute the homology groups $H_k(\mathbb{P}^1(\mathbb{C}), \mathbb{Z})$ for $k \in \mathbb{N}$.

4. Compute the homology groups $H_k(\mathbb{P}^2(\mathbb{C}), \mathbb{Z})$ for $k \in \mathbb{N}$.

5. Embed $\mathbb{P}^1(\mathbb{C})$ into $\mathbb{P}^2(\mathbb{C})$ as a line and show that the associated natural map $H_2(\mathbb{P}^1(\mathbb{C}), \mathbb{Z}) \to H_2(\mathbb{P}^2(\mathbb{C}), \mathbb{Z})$ is an isomorphism.

6. Let $C$ be a smooth (i.e., nonsingular) conic in $\mathbb{P}^2(\mathbb{C})$. Show that $C$ is homeomorphic to $S^2$. (Hint: project $C$ onto a line from a point in $C$.)

7. Compute the image of $H_2(C, \mathbb{Z})$ into $H_2(\mathbb{P}^2(\mathbb{C}), \mathbb{Z})$ under the natural map.
6 Differential Equations

The nonlinear Schrödinger equation
Consider the nonlinear Schrödinger equation

\[ iu_t + u_{xx} + 2|u|^2u = 0, \]  

(6.1)

where \( u(x, t) \) is a complex function of the two real variables \( x, t \), rapidly decreasing at infinity, that is, \( \lim_{|x| \to \infty} x^n u(x, t) = 0, \forall n \geq 0 \).

1. Show that, for a solution \( u = u(x, t) \) of eq. (6.1), the Hamiltonian and the momentum

\[ H = \int_{-\infty}^{+\infty} (|u_x|^2 - |u|^4)dx, \quad P = \frac{i}{2} \int_{-\infty}^{+\infty} (u \, u_x - uu_x^*)dx \]

are independent of time. Here, \( u^* \) is the complex conjugate of \( u(x, t) \). (Hint: use integration by parts).

2. Show that the equation is invariant under the following Galilean transformation

\[ \xi = x + vt \]
\[ s = t \]
\[ w(\xi, s) = u(x, t)e^{\frac{iv}{2}(x+vt)}, \]

that is, the function \( w(\xi, s) \) satisfies the equation

\[ iw_s + w_{\xi\xi} + 2|w|^2w = 0 \]

of the same form. Here \( v \) is an arbitrary real number.

3. Prove the existence of a solution rapidly decreasing at infinity of the form

\[ u(x, t) = c(x)e^{i\lambda}, \quad \lambda \text{ real positive constant, } c(x) \text{ real,} \]

(6.2)

observing that the equation for \( c(x) \) is a one-dimensional motion in a quartic potential.

4. Check that the solution of the form (6.2) has momentum \( P = 0 \). Using the Galilean invariance, write the corresponding solution having \( P \neq 0 \).