SISSA ISAS

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Mathematical Physics Sector

Entrance examination 2002/2003 – October Session

The candidate is asked to solve at least one problem among the following.

1 Quantum Mechanics

A Harmonic oscillator in an electric field

Consider the one–dimensional quantum harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

The particle has also a charge e. Switching on a constant electric field $\mathcal{E}$, the Hamiltonian gets an additional potential term and becomes

$$H'_- = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - e\mathcal{E}x \tag{1.1}$$

1. Using the well-known expression of the energy spectrum of the harmonic oscillator, determine the spectrum of the new Hamiltonian.

2. Prove that the two operators H'_- and H'_+ (obtained by changing $\mathcal{E} \to -\mathcal{E}$ in eq.(1.1) are unitarily equivalent in $\mathcal{L}_2(-\infty, \infty)$.

3. Determine the value of $\mathcal{E}$ such that the energy of the first excited state $|\,\psi'_1\rangle$ of H'_- coincides with the energy of ground state $|\,\psi_0\rangle$ of H. Which observable permits to distinguish $|\,\psi'_1\rangle$ from $|\,\psi_0\rangle$?

2 Linear algebra

Linear operators and matrices

Let $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator acting in the n-dimensional real space $\mathbb{R}^n$. Denote

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

the matrix of the linear operator with respect to some basis $e_1, e_2, \dots, e_n$ in the space $\mathbb{R}^n$, i.e.

$$\mathcal{A}e_j = \sum_{i=1}^{n} a_{ij}e_i, \quad j = 1, 2, \dots, n.$$

Denote $P_{\mathcal{A}}(t)$ the characteristic polynomial of the operator $\mathcal{A}$,

$$P_{\mathcal{A}}(t) = \det(A - t\,I).$$

Here

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

is the identity matrix.

1. Considering the complex numbers $\mathbb{C}$ as the two-dimensional real space $\mathbb{R}^2$, introduce the linear operator of multiplication by a given complex number λ,

$$\mathcal{A}\,z = \lambda z, \quad z \in \mathbb{C}.$$

Compute the characteristic polynomial $P_{\mathcal{A}}(t)$.

2. Let

$$\mathcal{A}, \mathcal{B} : \mathbb{R}^2 \to \mathbb{R}^2$$

be two linear operators acting in the two-dimensional real space satisfying

$$\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}.$$

Prove the existence of such a basis e_1, e_2 that the matrices A, B of the linear operators take one of the following forms:

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \tag{2.1}$$

or

$$A = \begin{pmatrix} a_1 & -a_2 \\ a_2 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & -b_2 \\ b_2 & b_1 \end{pmatrix} \tag{2.2}$$

or

$$A = \begin{pmatrix} a_1 & 1 \\ 0 & a_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_2 \\ 0 & b_1 \end{pmatrix} \tag{2.3}$$

for some real numbers a_1, a_2, b_1, b_2.

3. Let two linear operators

$$\mathcal{A}, \mathcal{B} : \mathbb{R}^3 \to \mathbb{R}^3$$

acting in three-dimensional real space satisfy the following equation

$$\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} = \mathcal{B}.$$

Compute the characteristic polynomial $P_{\mathcal{B}}(t)$.

3 Classical mechanics

A Classical Particle in a magnetic field
Consider the system defined in the Euclidean space $\mathbb{R}^n$ by the Lagrangian

$$L_{\mathbf{A}} = \frac{m}{2}\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} + \frac{q}{c}\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}) - V(\mathbf{x}), \tag{3.1}$$

where $\mathbf{x} = (x_1, \dots, x_n)$ are Cartesian coordinates, $\mathbf{A}(\mathbf{x})$ is the vector potential, $V(\mathbf{x})$ is the scalar potential, and $\dot{\mathbf{x}}$ denotes the velocity vector.
1) Prove that the Euler Lagrange equations associated with $L_{\mathbf{A}}$ are invariant under the gauge transformation

$$A(\mathbf{x}) \to A'(\mathbf{x}) = A(\mathbf{x}) + \operatorname{grad}\Phi(\mathbf{x}) \tag{3.2}$$

2) Prove that, in the Hamiltonian formalism, the transformation (3.2) yields a canonical transformation.
3) Consider the case of $\mathbb{R}^3$ endowed with Cartesian coordinates x, y, z. Use the above arguments to prove complete integrability (that is, to find an integral of the motion in addition to the obvious two) of the system with Lagrangian

$$L_B = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{q}{c}By\dot{x} - V(\rho), \tag{3.3}$$

where B is a constant and $\rho = \sqrt{x^2 + y^2}$.
(Hint: compute first the magnetic field associated with the vector potential $\mathbf{A} = [-By, 0, 0]$.*)*
4) Consider the Hamiltonian of the system (3.3), written in the appropriate gauge, and discuss qualitatively the resulting one dimensional radial motion, assuming $V(\rho) = \log\rho$, for generic values of the constants of the motion.

4 Statistical Mechanics

One-dimensional random walk

At discrete units of time ($t = 1, 2, 3, \cdots$), a particle moves along a one dimensional lattice. Suppose it starts at the origin and that at each step it moves either one lattice site to the right or to the left, each with probability 1/2.

For $n \geq 1$, let u_n be the probability that the particle returns to the origin at time $t = n$ and p_n the probability that the first return to the origin occurs at $t = n$. For convenience we take $p_0 = 0$ and $u_0 = 1$. Clearly $p_n = u_n = 0$ whenever n is odd.

1. Show that for $n \geq 1$, the two probabilities are related as

$$u_n = p_0 u_n + p_1 u_{n-1} + p_2 u_{n-2} + \cdots + p_n u_0 \qquad (4.1)$$

2. Prove that

$$u_{2n} = \frac{1}{2^{2n}} \binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \qquad (4.2)$$

3. Show that the particle will sooner or later return to its initial position, i.e.

$$p_2 + p_4 + p_6 + \cdots = 1$$

Hint. Introduce the generating functions

$$F(x) = \sum_{n=0}^{\infty} p_n x^n, \quad U(x) = \sum_{n=0}^{\infty} u_n x^n$$

and show that eq.(4.1) leads to the relation $U(x) = 1 + F(x)U(x)$. Use this relation and the u_{2n} given by eq.(4.2) to determine $F(x)$. It is useful to remind that

$$(1+x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \cdots + \frac{q(q-1)\ldots(q-k+1)}{k!}x^k + \cdots$$

5 Geometry

Projective spaces

Let $\mathbb{P}^n(\mathbb{C})$ be the n-dimensional complex projective space ($n \geq 1$).

1. Prove that $\mathbb{P}^1(\mathbb{C})$ is homeomorphic to the real 2-dimensional sphere S^2.
2. Show that $\mathbb{P}^n(\mathbb{C})$ contains a copy of $\mathbb{C}^n$ in a such a way that the complement is isomorphic to $\mathbb{P}^{n-1}(\mathbb{C})$.
3. Compute the homology groups $H_k(\mathbb{P}^1(\mathbb{C}), \mathbb{Z})$ for $k \in \mathbb{N}$.
4. Compute the homology groups $H_k(\mathbb{P}^2(\mathbb{C}), \mathbb{Z})$ for $k \in \mathbb{N}$.
5. Embed $\mathbb{P}^1(\mathbb{C})$ into $\mathbb{P}^2(\mathbb{C})$ as a line and show that the associated natural map $H_2(\mathbb{P}^1(\mathbb{C}), \mathbb{Z}) \to H_2(\mathbb{P}^2(\mathbb{C}), \mathbb{Z})$ is an isomorphism.
6. Let C be a smooth (i.e., nonsingular) conic in $\mathbb{P}^2(\mathbb{C})$. Show that C is homeomorphic to S^2. *(Hint: project C onto a line from a point in C.)*
7. Compute the image of $H_2(C, \mathbb{Z})$ into $H_2(\mathbb{P}^2(\mathbb{C}), \mathbb{Z})$ under the natural map.

6 Differential Equations

The nonlinear Schrödinger equation
Consider the nonlinear Schrödinger equation

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{6.1}$$

where $u(x,t)$ is a complex function of the two real variables x, t, rapidly decreasing at infinity, that is, $\lim_{|x|\to\infty} x^n u(x,t) = 0$, $\forall\, n \geq 0$.

1. Show that, for a solution $u = u(x,t)$ of eq. (6.1), the Hamiltonian and the momentum

   $$H = \int_{-\infty}^{+\infty} (|u_x|^2 - |u|^4)dx, \quad P = \frac{i}{2}\int_{-\infty}^{+\infty} (u^* u_x - u u_x^*)dx$$

   are independent of time. Here, u^* is the complex conjugate of $u(x,t)$. *(Hint: use integration by parts)*.

2. Show that the equation is invariant under the following Galilean transformation

   $$\begin{aligned} &\xi = x + vt \\ &s = t \\ &w(\xi, s) = u(x,t)e^{\frac{iv}{2}(x+\frac{v}{2}t)}, \end{aligned}$$

   that is, the function $w(\xi, s)$ satisfies the equation

   $$iw_s + w_{\xi\xi} + 2|w|^2 w = 0$$

   of the same form. Here v is an arbitrary real number.

3. Prove the existence of a solution rapidly decreasing at infinity of the form

   $$u(x,t) = c(x)e^{i\lambda t}, \quad \lambda \text{ real positive constant, } c(x) \text{ real}, \tag{6.2}$$

   observing that the equation for $c(x)$ is a one-dimensional motion in a quartic potential.

4. Check that the solution of the form (6.2) has momentum $P = 0$. Using the Galilean invariance, write the corresponding solution having $P \neq 0$.