International School for Advanced Studies — Trieste

Entrance Examinations 2003/2004 — October Session

PhD Programme in Geometry

All four texts consist of a short essay and an exercise. The candidate should at least complete one exercise with the related essay. A single complete exercise solved with all details is better than several incomplete exercises. Please note that only one essay will be evaluated.

You can write your test either in English or Italian.

1. Essay: Introduce and discuss the notions of vector field over a differentiable manifold, of integral curve of a vector field, and of critical point of a vector field.

Exercise:

Let M be the open submanifold of $\mathbb{R}^2$ given by the condition $x \neq 0$, and let X be the vector field on M

$$X = (x-1)\frac{\partial}{\partial x} + \frac{1-x}{x^2}\frac{\partial}{\partial y}.$$

a. Find the complete family of integral curves of X.

b. Determine the integral curve of X through the point $(1, 0)$.

c. Is there an integral curve of X that goes through both points $(2, 0)$ and $(4, 0)$?

d. Discuss the stability of the critical points of X.

2. Essay: A topological space X is said to be compact if every open cover has a finite subcover (sometimes the term “quasi-compact” is used). Discuss the main properties of compact topological spaces.

Exercise:

Consider the following statements about a topological space X.

A. X is compact.

B. For every topological space Y, the projection $p : X \times Y \to Y$ is closed (i.e., for every closed subset $C \subset X \times Y$, the subset $p(C)$ of Y is closed).

C. The projection $q \colon X \times Z \to Z$ is closed for every finite discrete topological space Z.

D. The projection $p : X \times \mathbb{R} \to \mathbb{R}$ is closed.

Then:

a. Prove that A implies B.

b. Show that C does not imply A in general.

c. Show that if X is second countable, then D implies A.

Please turn over

3. Essay: Discuss the diagonalizability of a a linear operator on a finite-dimensional complex vector space.

Exercise: Let V be a 3-dimensional vector space over the complex numbers $\mathbb{C}$, let W be the space of $\mathbb{C}$-linear operators $V \to V$, and let U be the subset of diagonalizable operators.

a. Prove that U is neither open nor closed but contains an open subset dense in W.

b. Define $e : W \to W$ by

$$e(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Prove that e is well defined and continuous.

c. Prove that for every invertible operator M one has

$$e(MAM^{-1}) = Me(A)M^{-1}.$$

d. Prove that for any $A \in W$ one has $\det e(A) = e^{\operatorname{tr}(A)}$.

4. Essay: Recall how a linear system on a complex algebraic variety X can define an embedding of X into a projective space. Recall the notion of complete intersection subvariety.

Exercise: Consider in the complex projective plane $\mathbb{P}^2$ the linear system L formed by the conics that pass through a fixed point.

a. Show that L defines an embedding of $\mathbb{P}^2$ blown up at a point into $\mathbb{P}^4$.

b. Let S be the image of this embedding. Check that S is not a complete intersection and find the minimal degrees of the generators of the ideal cutting S.

c. Consider a generic projection of S into $\mathbb{P}^3$. Show that its image is singular.