International School for Advanced Studies — Trieste

Entrance Examinations 2003/2004 — April Session

PhD Programme in Geometry

All texts consist of a short essay and an exercise. The candidate should at least complete one exercise with the related essay. A single complete exercise solved with all details is better than several incomplete exercises. Please note that only one essay will be evaluated.

1. Essay: Define the Alexandroff (also called one-point) compactification of a Hausdorff topological space; state its main properties and prove at least one of them.

Exercise:

a. Let X be $S^1 \times S^1$ minus one point, and let Y be S^2 minus 3 points. Find the Alexandroff compactification of X and of Y.

b. Prove that X is not homeomorphic to Y.

c. Compute the fundamental groups of X and of Y.

d. Prove that X is homotopically equivalent to Y.

2. Essay: Recall the definition of Riemannian manifold, proving that any differentiable manifold admits Riemannian metrics. Recall the local existence and uniqueness theorem for geodesics.

Exercise: Prove that a 1-dimensional connected manifold is diffeomorphic either to the real line $\mathbb{R}$ or to the circle S^1.

3. Essay: Recall the definition of hypersurface in affine and projective space (over the complex numbers). State the main properties of the hypersurfaces, proving at least one of them.

Exercise:

a. Let X be an irreducible hypersurface of degree d with a point of multiplicity $d-1$. Prove that X is rational.

b. Let X be an affine hypersurface of equation $F(x_1, \ldots, x_n) = 0$, where the polynomial F is a sum of two nonzero homogeneous polynomials F_{d-1} and F_d of degree $d-1$ and d respectively, having no common factors. Prove that X is irreducible and rational.

c. Denote by E_0, E_1, E_2, E_3 the fundamental points of a projective frame in $\mathbb{P}^3$. Let S be an algebraic surface that contains the three lines $\overline{E_0E_1}$, $\overline{E_0E_2}$, $\overline{E_0E_3}$. Prove that S is singular.

4. Essay: Briefly recall the notion of principal directions, lines of curvature, Gaussian and average curvature and umbilical point for a surface embedded in $\mathbb{R}^n$.

Exercise: Consider the surface $x^2 + y^2 = z^2 + 1$. Determine:

a. its umbilical points (if any);

b. its curvature lines;

c. the Gaussian and average curvature at every point.

d. Show that for any surface of revolution, the lines of curvature are the parallels and the meridians.