All texts consist of a short essay and an exercise. The candidate should at least complete one exercise with the related essay. A single complete exercise solved with all details is better than several incomplete exercises. Please note that only one essay will be evaluated.

1. **Essay:** Define the Alexandroff (also called one-point) compactification of a Hausdorff topological space; state its main properties and prove at least one of them.

   **Exercise:**
   a. Let $X$ be $S^1 \times S^1$ minus one point, and let $Y$ be $S^2$ minus 3 points. Find the Alexandroff compactification of $X$ and of $Y$.
   b. Prove that $X$ is not homeomorphic to $Y$.
   c. Compute the fundamental groups of $X$ and of $Y$.
   d. Prove that $X$ is homotopically equivalent to $Y$.

2. **Essay:** Recall the definition of Riemannian manifold, proving that any differentiable manifold admits Riemannian metrics. Recall the local existence and uniqueness theorem for geodesics.

   **Exercise:** Prove that a 1-dimensional connected manifold is diffeomorphic either to the real line $\mathbb{R}$ or to the circle $S^1$.

3. **Essay:** Recall the definition of hypersurface in affine and projective space (over the complex numbers). State the main properties of the hypersurfaces, proving at least one of them.

   **Exercise:**
   a. Let $X$ be an irreducible hypersurface of degree $d$ with a point of multiplicity $d-1$. Prove that $X$ is rational.
   b. Let $X$ be an affine hypersurface of equation $F(x_1, \ldots, x_n) = 0$, where the polynomial $F$ is a sum of two nonzero homogeneous polynomials $F_{d-1}$ and $F_d$ of degree $d-1$ and $d$ respectively, having no common factors. Prove that $X$ is irreducible and rational.
   c. Denote by $E_0, E_1, E_2, E_3$ the fundamental points of a projective frame in $\mathbb{P}^3$. Let $S$ be an algebraic surface that contains the three lines $E_0E_1, E_0E_2, E_0E_3$. Prove that $S$ is singular.

4. **Essay:** Briefly recall the notion of principal directions, lines of curvature, Gaussian and average curvature and umbilical point for a surface embedded in $\mathbb{R}^n$.

   **Exercise:** Consider the surface $x^2 + y^2 = z^2 + 1$. Determine:
   a. its umbilical points (if any);
   b. its curvature lines;
   c. the Gaussian and average curvature at every point.
   d. Show that for any surface of revolution, the lines of curvature are the parallels and the meridians.