SISSA ISAS

SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

I-34014 Trieste ITALY - Via Beirut 4 - Tel. [+]39-40-37871 - Telex:460269 SISSA I - Fax: [+]39-40-3787528

SISSA Mathematical Physics Sector.
Entrance examination. Spring Session 2003.

The candidate is asked to solve at least one of the following exercises.

1 Classical Mechanics

Consider the motion of a unit mass particle in the plane, with central attractive force.

1) Briefly discuss the conservation laws and the qualitative aspects of the radial motion.

2) Let $-\frac{\mu}{4r^4}$ the potential per unit mass. Write the Hamiltonian in polar coordinates. Show that the equation of the orbit $r = r(\theta)$ for fixed values of the energy E and angular momentum L is

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{2E}{L^2}r^4 - r^2 + \frac{\mu}{2L^2}.$$

Integrate the equations for the values of the energy $E = 0$ and $E = \frac{L^4}{4\mu}$ observing in this latter case that there is more than one orbit.

3) Discuss qualitatively the trajectories in the cases

i)$E < 0$,

ii)$0 < E < \frac{L^4}{4\mu}$,

iii) $E > \frac{L^4}{4\mu}$,

specifying whether the orbit is bounded or unbounded and passes through the centre of attraction.

2 Quantum Mechanics

Consider a quantum mechanical particle with energy E in the one-dimensional potential $V(x)$ which takes the constant value V_0 for $0 < x < a$ and vanishes identically outside this interval.

1. Take $V_0 < E < 0$.

a) Discuss the conditions that the wave-function has to satisfy at the points $x = 0$ and $x = a$.

b) Determine the quantization condition for the energy levels.

c) Determine the minimal number of levels that can be obtained varying the parameters a and V_0 ($aV_0 \neq 0$).

d) Derive from the quantization condition the energy spectrum in the case $V_0 = -\infty$.

2. Take $0 < E < V_0$ and suppose that the particle travels from left to right.

e) Determine the transmission amplitude of the particle through the barrier.

f) Determine the transmission amplitude corresponding to the limiting case $V(x) = g\delta(x)$ and interpret its singular point.

3 Symmetric Matrices and Quadratic Forms

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

be a symmetric matrix of the order n with all real entries $a_{ji} = a_{ij}$.

1. Briefly discuss relationships between the eigenvalues and eigenvectors of the matrix A and the constrained extrema of the quadratic form

$$q(x) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}x_i x_j, \quad x = (x_1, x_2, \dots, x_n)$$

on the unit sphere

$$x_1^2 + x_2^2 + \dots + x_n^2 = 1. \tag{1}$$

2. Let A, B be two real symmetric matrices. Assuming that the eigenvalues of the matrices A and B belong to the intervals $[\alpha_1, \alpha_2]$ and $[\beta_1, \beta_2]$ respectively, prove that the eigenvalues of the matrix $A + B$ belong to the interval $[\alpha_1 + \beta_1, \alpha_2 + \beta_2]$.

3. Let us consider the quadratic form

$$q(x) = 2 \sum_{k=1}^{n-1} x_k x_{k+1}. \tag{2}$$

a) Find the associated $n \times n$ symmetric matrix A.

b) Prove that the characteristic polynomial of this matrix

$$D_n(\lambda) = \det(\lambda\, I - A)$$

can be represented in the form

$$D_n(\lambda) = \frac{e^{(n+1)t} - e^{-(n+1)t}}{e^t - e^{-t}}$$

after the substitution

$$\lambda = e^t + e^{-t}.$$

Hint: derive the recursion relation for the characteristic polynomials

$$D_n(\lambda) - \lambda\, D_{n-1}(\lambda) + D_{n-2}(\lambda) = 0.$$

c) Find the constrained extrema of the quadratic form (2) on the sphere (1).