INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Admission to the PhD programme in Geometry
Academic Year 2004/2005

Written examination of 14/10/2004

The candidate should solve at least one exercise.

Exercise 1. Let $f : X \to Y$ be a surjective C^∞ map between C^∞ manifolds. Let us recall that f is said to be a *submersion* if for all $x \in X$ the differential $df(x)$ is surjective. For each of the following statements, prove it if it is true, and give a counterexample if it is false.

1. f is a submersion.
2. If f is a submersion, for any $y \in Y$ there exist a neighbourhood U of y in Y and a C^∞ map $s : U \to X$ such that $f \circ s = \mathrm{id}_U$.
3. Let Z be a C^∞ manifold, and let $g : Y \to Z$ be a map. If $g \circ f$ is C^∞ then g is C^∞ as well.
4. Let Z be a C^∞ manifold, and let $g : Y \to Z$ be a map. If f is a submersion, then $g \circ f$ is C^∞ is and only if g is C^∞.

Exercise 2. Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^3(\mathbb{C})$ be the morphism whose expression in homogeneous coordinates is $f(s,t) = (s^3, t^3, st^2 + s^2t, st^2 - s^2t)$. Let C be its image.

1. Prove that $f : \mathbb{P}^1 \to C$ is an isomorphism.
2. Find the explicit generators of the ideal I which cuts C in $\mathbb{P}^3$.
3. Say if I can be generated by two elements.
4. Let $H \subset \mathbb{P}^3$ be a hyperplane and p a point in $\mathbb{P}^3$ outside H. Let $\pi : \mathbb{P}^3 \setminus p \to H$ be the projection with centre in p. Say for what choices of p and H the rational map g from C to H defined by $\pi|_C$ is an isomorphism onto its image.

Exercise 3. Recall that an n-dimensional topological manifold is a Hausdorff second countable topological spaces locally homeomorphic to $\mathbb{R}^n$. Answer each of the following points, fully motivating every statement.

1. Say if the union of a circumference in the cartesian plane with a secant line, equipped with the relative topology, is a topological manifold.
2. Prove that the n-dimensional sphere

   $$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$$

   admits a structure of topological manifold.

3. The group $\mathbb{Z}_2$ acts on S^n according to the rule $gx = -x$, where g is the generator of $\mathbb{Z}_2$. Denoting $\mathbb{RP}^n = S^n/\mathbb{Z}_2$, prove that $\mathbb{RP}^n$ admits a structure of topological manifold.

4. Prove that $\mathbb{RP}^1 \simeq S^1$.

Exercise 4. Let

$$X = x\frac{\partial}{\partial y} - \frac{\partial}{\partial x}$$

$$Y = e^{P(x,y)}\,\frac{\partial}{\partial y} + e^{-x}\,\frac{\partial}{\partial z}$$

be vector fields in $\mathbb{R}^3$, with $P(x, y)$ a polynomial (x, y, z are cartesian coordinates in $\mathbb{R}^3$).

1. After choosing $P(x, y) = 0$ say if there exists a surface in $\mathbb{R}^3$ which is tangent to X and Y at every point.

2. Determine a polynomial P such that the commutation rule $[X, Y] = Y$ holds.

3. Having chosen the polynomial P in this way, say if there are surfaces in $\mathbb{R}^3$ that are tangent to X and Y at every point. If true, find such a surface passing through the origin.

4. Again with choice of the polynomial P, find, if possible, vector fields X' and Y' such that $[X', Y'] = 0$ which at every point in $\mathbb{R}^3$ generate the same vector space than X and Y.