SISSA Mathematical Physics Sector.
Entrance examination. 2004 Session .

The candidate is asked to solve at least one of the following exercises.

1 Quantum Mechanics

Consider a free quantum particle of mass m which lies on a circle of radius R. Its coordinate is q with $q \equiv q + 2\pi R$ and its lagrangian is given by

$$L = \frac{m}{2}\left(\frac{dq}{dt}\right)^2$$

1. Determine the energy eigenfunctions, the energy levels and their degeneracy.
2. Change the lagrangian according to

   $$L = \frac{m}{2}\left(\frac{dq}{dt}\right)^2 - \hat{\theta}\,\frac{dq}{dt}$$

   Write the corresponding Hamiltonian and determine the new quantum energy levels. Study their degeneracy as a function of the dimensionless parameter $\theta = \hat{\theta}R/\hbar$ in the interval $0 \leq \theta \leq 1$.
3. Prove that the spectrum is invariant for $\theta \to \theta \pm k$, where k is any integer number.
4. Take $\theta = 1/3$ and suppose that at $t = 0$ system is in the physical state described by the wave function $\psi(q) = \cos\frac{q}{R}$. Determine the minimal time required to the system for coming back to the same physical state.

2 Analytical Mechanics

Consider, in the configuration space $\mathbb{R}^2$ the Lagrangian

$$L = T - V = \frac{1}{2}(\dot{x}^2 + (1 + x^2)\dot{y}^2) - V(x), \tag{2.1}$$

where $(x, y) \in \mathbb{R}^2$, $V(x) \in C^\infty(\mathbb{R})$.

1) Compute the Hamiltonian H corresponding to L. Find the constants of the motion, and show that the stationary Hamilton-Jacobi equation $H = E$ is separable in the coordinates x, y, i.e., it admits a solution of the form $S(x, y) = S_1(x) + S_2(y)$.

2) Show that the Euler-Lagrange equations defined by (**??**) are equivalent to a system of the form:

$$\dot{y} = \phi(x; c) \tag{2.2}$$

$$\ddot{x} = \psi(x; c), \tag{2.3}$$

for two suitable functions ϕ, ψ and a constant c.

Show that equation (**??**) is an Euler-Lagrange equation for a system with one degree of freedom and compute its Lagrangian L_{red}.

3) Set $\dot{y}$ equal to a constant (say, $\dot{y} = \omega$) and

$$V(x) = \frac{x^4}{4} - a\frac{x^2}{2}$$

in L, and qualitatively study the types of open and closed orbits in the one-dimensional system so obtained, for all values of the parameters a, ω.

4) Study the equilibrium points of the system (**??**) for the potential

$$V(x, y) = \frac{x^4}{4} - a\frac{x^2}{2} + \frac{1}{2}(x - y)^2$$

Determine, in particular, the frequencies of the small oscillations around stable equilibrium for all $a \neq 0$.

3 Theory of Functions

1. For a given complex number $z \neq 0$ describe all solutions w to the equation

$$w^n = z.$$

2. Consider the analytic function $f(z)$ of the complex argument z defined in the neighborhood $|z-1| < \frac{1}{2}$ of the point $z = 1$ by

$$[f(z)]^n = z, \quad f(1) = 1.$$

Denote $g(z)$ the result of the counter-clockwise analytic continuation of the function $f(z)$ along the path

$$z = e^{i\phi}, \quad 0 \leq \phi \leq 2\pi$$

to the same neighborhood of $z = 1$. Evaluate $g(1)$.

3. Prove that there exist exactly two functions $F_1(z)$ and $F_2(z)$ analytic in the upper half-plane $\operatorname{Im} z > 0$ and satisfying

$$[F_{1,2}(z)]^2 = z(z-2)(z-3)(z-5)(z-9). \qquad (1).$$

4. a) Prove existence of the limits

$$F_k(x) := \lim_{z \to x, \ \operatorname{Im} z > 0} F_k(z), \quad k = 1, 2 \qquad (2)$$

for any real x ($F_k(x)$ in the left hand side of the last equation is *defined* as the value of the limit).
b) Let $F_1(z)$ be the solution to (1) such that the limits (2) are positive for real x satisfying $0 < x < 2$. Evaluate the above limits $F_1(1)$, $F_1(4)$, $F_1(10)$ and $F_2(1)$, $F_2(4)$, $F_2(10)$ of the functions $F_1(z)$, $F_2(z)$ defined in (1).

4 Theory of Operators

Consider the linear operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$T(f)(x) = \int_{-\infty}^{+\infty} K(x,y)\, f(y)\, dy, \tag{4.1}$$

with

$$K(x,y) = \exp\left[-\frac{1}{4}(x^2+y^2) + \frac{1}{2}Jxy\right]. \tag{4.2}$$

Here J is a real parameter.

1. Recall that an integral operator of the form (**??**) with a real valued kernel $K(x,y)$ is a Hilbert-Schmidt operator if

   $$||T||_{HS}^2 := \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K(x,y)K(y,x)\, dx\, dy \, < \infty. \tag{4.3}$$

   The number $||T||_{HS}$ so obtained is called the norm of the Hilbert-Schmidt operator.

   Determine all values of the parameter J for which the operator T defined by (**??**), (**??**) is a Hilbert-Schmidt operator and compute its norm.

   Useful formula:

   $$\int_{-\infty}^{+\infty} \exp[-\alpha x^2 + bx]\, dx \; = \; \sqrt{\frac{\pi}{\alpha}}\, \exp\left[\frac{b^2}{4\alpha}\right]$$

2. Let J be any number such that $||T||_{HS} < \infty$. Introduce the linear operator $\mathbf{A}(a)$ by

   $$\mathbf{A}(a) = a\, x + \frac{d}{dx}$$

   a) Find the value $a_0 > 0$ of the parameter a such that $\mathbf{A}_0 := \mathbf{A}(a_0)$ satisfies

   $$\mathbf{A}_0 T = \xi\, T \mathbf{A}_0$$

   for some real ξ. Compute also the value of ξ.

   b) Observe that $\mathbf{A}_0^\dagger = a_0\, x - \frac{d}{dx}$ is the operator adjoint to $\mathbf{A}_0$ and prove that

   $$T\mathbf{A}_0^\dagger = \xi\, \mathbf{A}_0^\dagger T.$$

3. Prove that for any J for which the operator T is Hilbert-Schmidt, its eigenfunction $\psi_0(x)$ corresponding to its highest eigenvalue λ_0 must satisfy

$$\mathbf{A}_0\psi_0 = 0.$$

Compute $\psi_0(x)$ and λ_0.

4. Determine all the other eigenvalues $\lambda_1, \lambda_2, \ldots$ of the operator T.

Hint: Use the identity

$$\sum_{n=0}^{\infty} \lambda_n^2 = ||T||_{HS}^2$$

valid for the eigenvalues of any Hilbert-Schmidt operator T.