INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Trieste

Admission to the PhD programme in Geometry
Academic Year 2005/2006

Written examination of 26/9/2005

The candidate should solve at least one exercise.

1.

a) Sketch an outline of the theory of integration of differential forms on compact oriented manifolds.

b) Let
$$\omega = (4x^2 + z^2)\, dy \wedge dz + 4xy\, dz \wedge dx + z^2\, dx \wedge dy$$
be a differential 2-form on $\mathbb{R}^3$. Let S be the portion of the "right" side of the surface having equation $4x^2 - y^2 = a^2$ which is contained in the cone $x = \sqrt{y^2 + z^2}$. Here a is a positive constant.

Compute the integral
$$\int_S \omega.$$

2.

a) State the Poincaré Lemma in de Rham cohomology, and sketch a proof.

b) Let
$$\eta = (\cos y + y\ \cos x)\, dx + (\sin x - x\ \sin y)\, dy$$
be a differential 1-form on $\mathbb{R}^2$. Show that η is exact, and find a function $u(x, y)$ such that $\eta = du$.

c) Make an example of an open subset $\Omega \subset \mathbb{R}^2$, and a differential 1-form η on it, such that η is closed but not exact. Compute the fundamental group of Ω.

d) Give an example of a differentiable manifold whose first de Rham cohomology group is zero (i.e., such that all closed 1-forms on it are exact), but is not simply connected.

3. Give an example of a "bump" function, i.e., a C^∞ function on $\mathbb{R}^n$ such that $f(0) > 0$ and $f(p) = 0$ if $\|p\| \geq 1$ and illustrate some uses of bump functions in differential geometry.

Prove at least 3 of the following results.

(a) Let n be a positive integer. There exists a diffeomorphiism $f: \mathbb{R}^n \to \mathbb{R}^n$ su that $f(0) = (1, 0, \ldots, 0)$ and $f(p) = p$ if $||p|| \geq 2$.

(b) Using (a), show that for all $a > 0$ and all $q \in \mathbb{R}^n$ such that $||q|| < a$, there exists a diffeomorphism $g: \mathbb{R}^n \to \mathbb{R}^n$ such that $f(0) = q$ and $f(p) = p$ if $||p|| \geq a$.

(c) Let X be a differentiable manifold of dimension n, and $p_0 \in X$. Let $X(p_0)$ be the set of points $q \in X$ such that there exists a diffeomorphism $f: X \to X$ with $f(p_0) = q$. Using (b), prove that $X(p_0)$ is open in X and nonempty.

(d) With the notation of point c) and using points (b) and (c), prove that if X is connected then $X(p_0) = X$.

4.

(a) State the definition of the Zariski topology for the affine complex space and discuss its main properties.

(b) Let V be the (affine) space formed by the $n \times m$ matrices with complex entries. Prove that the subset of V_r formed by matrices of rank $\leq r$ is closed in the Zariski topology, so that it is an affine variety. Find values of m, n, r such that V_r is a singular variety.

(c) Assume that $A \in V$ has an invertible minor B of dimension $r \times r$. Prove that A has rank r if and only if every minor B' of dimension $(r+1) \times (r+1)$ containing B is not invertible.

(d) Prove that $U_r = V_r - V_{r-1}$ is a smooth algebraic variety of dimension $nm - (n - r)(m - r)$. Hint: use point (c).