International School for Advanced Studies

Trieste

Admission to the PhD Programme in Geometry

Academic Year 2006/07

The candidate should solve at least one exercise.

1. Define the notion of homotopy between two continuous maps of topological spaces.

Classify all continuous maps $S^1 \to S^1$ up to homotopy, motivating your answer.

Let $f : S^1 \to S^1$ be a continuous map. Show that if f has no fixed point then it is homotopic to $v \mapsto -v$. Show that if there is no point $v \in S^1$ such that $f(v) = -v$ then f is homotopic to the identity.

2. All rings will be commutative with 1, and all ring homomorphisms will be unitary.

Let B be a domain, and $A \subset B$ a subdomain. Define the integral closure of A in B, and prove that it is a ring.

For each of these domains, determine their integral closure in their quotient field: the integers; $\mathbb{C}[x, y]/x^3 - y^2$.

Prove that if a domain A is integrally closed (i.e., coincides with its integral closure inside its quotient field) and G is a group of automorphisms of A, then

$$A^G := \{a \in A \,|\, g(a) = a \ \forall g \in G\}$$

is also integrally closed. Prove that $\mathbb{C}[x, y, z]/xy - z^2$ is integrally closed.

3. Let K be a field, and V a finite-dimensional K-vector space. Define the notion of symmetric bilinear form on V. Define the symmetric matrix associated to a bilinear form and a basis of V. Explain what it means for a symmetric bilinear form to be diagonalizable and to be nondegenerate.

Prove that if $\operatorname{char} K \neq 2$ every symmetric bilinear form can be diagonalized.

Consider the following bilinear form on $(\mathbb{F}_2)^2$:

$$f((x_1, x_2), (y_1, y_2)) = x_1 y_2 + x_2 y_1.$$

Is it diagonalizable? Is it nondegenerate?

Prove that if $\operatorname{char} K \neq 2$ every symmetric bilinear form can be represented by a matrix having on the diagonal only zeroes and elements in a choice of representatives for K/K^2.

4. Define the degree of an (irreducible projective) curve in complex projective n-dimensional space.

Prove that the image of the map $f : \mathbb{P}^1 \to \mathbb{P}^3$ given by $f(x, y) = (x^3, y^3, x^2 y, x y^2)$ is a curve C_0 of degree 3. Find equations for C and prove that it is not a complete intersection.

Let C be a curve in $\mathbb{P}^n$ and $p \in C$ a smooth point; let C' be the closure in $\mathbb{P}^{n-1}$ of the image of $C \setminus p$ via the projection from p. Prove that $\deg C' = \deg C - 1$, and that the rational map $C \to C'$ is a morphism. Give an example where p is not a smooth point and at least one of these statements is false.

Prove that every degree 3 curve in $\mathbb{P}^3$ is either isomorphic to C_0 after a coordinate change in $\mathbb{P}^3$ or is contained in a plane.

5. Let X be a smooth complex projective variety. Define the group of divisors $\mathrm{Div}(X)$ and the group of isomorphism classes of line bundles $\mathrm{Pic}(X)$. Describe the relationship between $\mathrm{Div}(X)$ and $\mathrm{Pic}(X)$.

Prove that $\mathrm{Pic}(\mathbb{P}^n)$ is cyclic, generated by $\mathcal{O}(1)$. Prove that every automorphism of $\mathbb{P}^n$ is a linear transformation.

Define the canonical line bundle K_X on X. Prove that $K_{\mathbb{P}^n}$ is isomorphic to $\mathcal{O}(-n-1)$.

Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. Determine K_X. Show that if $d \neq n+1$, then every automorphism of X is induced by an automorphism of $\mathbb{P}^n$; prove that $\mathrm{Aut}(X)$ is a closed algebraic subgroup of $PGL(n)$.

Give an example where $d = n+1$ and X has an automorphism not induced by an automorphism of $\mathbb{P}^n$.

6. Let X be the subset of the real projective plane $\mathbb{P}^2(\mathbb{R})$ given in homogeneous coordinates (u, v, w) by the equation

$$v^2 w = u^3 - 3u^2 w + uw^2 \,.$$

a) Prove that X is a submanifold of $\mathbb{P}^2(\mathbb{R})$.
b) Prove that X is compact.

Introduce in the open subset U_2 where $w \neq 0$ affine coordinates $x = u/w$, $y = v/w$, and define the differential 1-form

$$\phi = \frac{dx}{y} \,.$$

c) Prove that ϕ is well defined in U_2.
d) Prove that ϕ may be extended to a nowhere vanishing 1-form on X.