

**Mathematical Physics Sector**  
**Entrance examination. 2006 Session**  
**Ph. D. in Mathematical Physics.**

The candidate is asked to solve at least one of the following exercises.

**Exercise 1.**

Given a function  $f(x)$  real analytic<sup>1</sup> at the point  $x = 0$  such that  $f(0) = 1$ , suppose that  $f(x)$  satisfies the equation

$$f\left(\frac{2x}{1+x^2}\right) = (1+x^2)f(x) \quad (0.1)$$

for an arbitrary sufficiently small  $x \in \mathbb{R}$ .

- 1) Prove that the equation (0.1) together with the normalization  $f(0) = 1$  determines the function  $f(x)$  uniquely
- 2) Prove that  $f(x)$  is an even function.
- 3) Expanding the function  $f(x)$  in a Taylor series

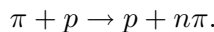
$$f(x) = 1 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

prove that

$$a_{2k+1} = 0, \quad a_{2k} = \frac{1}{2^k + 1}, \quad k = 0, 1, 2, \dots$$

**Exercise 2.**

- 1) Briefly introduce the dynamics of the collisions of two relativistic particles.
- 2) A pion with kinetic energy  $T=1200$  MeV interacts with a proton at rest, producing  $n$  pions via the reaction



What is the maximum number of pions that can be produced? (The rest energies of the proton and of the pion are 938 and 140 MeV respectively.)

**Exercise 3**

Let us consider a particle of mass  $m$  in a potential  $U = U(\mathbf{x})$ ,  $\mathbf{x} \in \mathbb{R}^3$  where

$$U(\mathbf{x}) = -\frac{k}{|\mathbf{x}|^\alpha}, \quad k > 0, \quad \alpha > 0.$$

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<sup>1</sup>Recall that a function  $f(x)$  is called real analytic at the point  $x = x_0 \in \mathbb{R}$  if  $f(x)$  can be represented as the sum of a power series

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots$$

convergent for  $|x - x_0| < \epsilon$  for some positive  $\epsilon$ .

1. Determine the Hamiltonian of the system in polar coordinates  $(r, \theta, \phi)$  and the first integrals.
2. Determine the effective potential  $W(r)$  and discuss when the orbits are bounded.
3. Determine the trajectory  $r = r(\theta)$ .
4. Determine the energy value  $E_0$  such that  $r(\theta) = \text{constant}$ .
5. For  $E$  close to  $E_0$ , determine perturbatively (let  $m = 1$ )

$$\Delta\theta = \int_{r_{min}}^{r_{max}} \frac{Mdr}{r^2 \sqrt{2(E - W(r))}}$$

where  $E$  and  $M$  are the energy and the angular momentum of the system.

6. For  $\alpha = 1$  show that all the bounded orbits are closed.

**Exercise 4.**

Let  $\mathbb{R}^{2n}$  be an Euclidean space of dimension  $2n$ , with Cartesian coordinates  $(p_1, \dots, p_n, q_1, \dots, q_n)$ . Consider the matrix

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where  $I$  is the rank  $n$  identity matrix; define the *symplectic product*  $[\cdot, \cdot]$  of two vectors  $\xi, \eta \in \mathbb{R}^{2n}$  as

$$[\xi, \eta] = (J\xi, \eta),$$

where  $(\cdot, \cdot)$  is the Euclidean scalar product in  $\mathbb{R}^{2n}$ .

1. Prove that  $[\cdot, \cdot]$  defines an antisymmetric bilinear form on  $\mathbb{R}^{2n}$ .
2. Show that if  $[\xi, \eta] = 0$  for all  $\xi \in \mathbb{R}^{2n}$ , then  $\eta = 0$ .
3. Two non zero vectors  $\xi$  and  $\eta$  are called anti-orthogonal if  $[\xi, \eta] = 0$ . Show that the set of all vectors anti-orthogonal to a given  $\eta \neq 0$  is a  $2n - 1$ -dimensional hyperplane in  $\mathbb{R}^{2n}$ .
4. A linear subspace  $V \subset \mathbb{R}^{2n}$  is called *isotropic* if the symplectic product of any two vectors in  $V$  is equal to zero. Prove that the dimension of an isotropic subspace  $V \subset \mathbb{R}^{2n}$  cannot exceed  $n$ .
5. Show that among the  $(2n)!/(n!)^2$   $n$ -dimensional coordinate subspaces of  $\mathbb{R}^{2n}$ , exactly  $2^n$  are isotropic. Determine these subspaces in the coordinates  $p_1, \dots, p_n, q_1, \dots, q_n$ .
6. Show that the linear maps  $S : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  such that

$$[S\xi, S\eta] = [\xi, \eta], \quad \forall \xi, \eta \in \mathbb{R}^{2n},$$

form a group. *Hint:* show that the matrix of the linear transformation  $S$  satisfies

$$S^t J S = J.$$

7. Given an eigenvalue  $\lambda$  of  $S$  prove that  $\lambda \neq 0$  and, moreover,  $1/\lambda$  is also an eigenvalue of  $S$ .