SISSA ISAS

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Mathematical Physics Sector
Entrance examination. 2006 Session
Ph. D. in Mathematical Physics.

The candidate is asked to solve at least one of the following exercises.

Exercise 1.
Given a function $f(x)$ real analytic[1] at the point $x = 0$ such that $f(0) = 1$, suppose that $f(x)$ satisfies the equation

$$f\left(\frac{2x}{1+x^2}\right) = (1+x^2)\, f(x) \tag{0.1}$$

for an arbitrary sufficiently small $x \in \mathbb{R}$.

1) Prove that the equation (0.1) together with the normalization $f(0) = 1$ determines the function $f(x)$ uniquely

2) Prove that $f(x)$ is an even function.

3) Expanding the function $f(x)$ in a Taylor series

$$f(x) = 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

prove that

$$a_{2k+1} = 0, \quad a_{2k} = \frac{1}{2k+1}, \quad k = 0,\, 1,\, 2, \dots.$$

Exercise 2.
1) Briefly introduce the dynamics of the collisions of two relativistic particles.
2) A pion witn kinetic energy T=1200 MeV interacts with a proton at rest, producing n pions via the reaction

$$\pi + p \to p + n\pi.$$

What is the maximun number of pions that can be produced? (The rest energies of the proton and of the pion are 938 and 140 MeV respectively.)

Exercise 3
Let us consider a particle of mass m in a potential $U = U(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^3$ where

$$U(\boldsymbol{x}) = -\frac{k}{|\boldsymbol{x}|^{\alpha}}, \quad k > 0, \quad \alpha > 0.$$

[1]Recall that a function $f(x)$ is called real analytic at the point $x = x_0 \in \mathbb{R}$ if $f(x)$ can be represented as the sum of a power series

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots$$

convergent for $|x - x_0| < \epsilon$ for some positive ϵ.

1. Determine the Hamiltonian of the system in polar coordinates (r, θ, ϕ) and the first integrals.
2. Determine the effective potential $W(r)$ and discuss when the orbits are bounded.
3. Determine the trajectory $r = r(\theta)$.
4. Determine the energy value E_0 such that $r(\theta)$ =constant.
5. For E close to E_0, determine pertubatively (let $m = 1$)

$$\Delta\theta = \int_{r_{min}}^{r_{max}} \frac{M dr}{r^2\sqrt{2(E - W(r))}}$$

   where E and M are the energy and the angular momentum of the system.
6. For $\alpha = 1$ show that all the bounded orbits are closed.

Exercise 4.
Let $\mathbb{R}^{2n}$ be an Euclidean space of dimension $2n$, with Cartesian coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$. Consider the matrix

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

where I is the rank n identity matrix; define the *symplectic product* $[\,.\,,\,.\,]$ of two vectors ξ, $\eta \in \mathbb{R}^{2n}$ as

$$[\xi, \eta] = (J\xi, \eta),$$

where $(\,.\,,.\,)$ is the Euclidean scalar product in $\mathbb{R}^{2n}$.

1. Prove that $[\,.\,,\,.\,]$ defines an antisymmetric bilinear form on $\mathbb{R}^{2n}$.
2. Show that if $[\xi, \eta] = 0$ for all $\xi \in \mathbb{R}^{2n}$, then $\eta = 0$.
3. Two non zero vectors ξ and η are called anti-orthogonal if $[\xi, \eta] = 0$. Show that the set of all vectors anti-orthogonal to a given $\eta \neq 0$ is a $2n - 1$-dimensional hyperplane in $\mathbb{R}^{2n}$.
4. A linear subspace $V \subset \mathbb{R}^{2n}$ is called *isotropic* if the symplectic product of any two vectors in V is equal to zero. Prove that the dimension of an isotropic subspace $V \subset \mathbb{R}^{2n}$ cannot exceed n.
5. Show that among the $(2n)!/(n!)^2$ n-dimensional coordinate subspaces of $\mathbb{R}^{2n}$, exactly 2^n are isotropic.
   Determine these subspaces in the coordinates $p_1, \dots, p_n, q_1, \dots, q_n$.
6. Show that the linear maps $S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that

$$[S\xi, S\eta] = [\xi, \eta], \quad \forall \xi, \eta \in \mathbb{R}^{2n},$$

   form a group. *Hint*: show that the matrix of the linear transformation S satisfies

$$S^t J S = J.$$

7. Given an eigenvalue λ of S prove that $\lambda \neq 0$ and, moreover, $1/\lambda$ is also an eigenvalue of S.