Mathematical Physics Sector
Entrance Examination, 2007 Session.
Ph.D. in Mathematical Physics

The candidate is asked to solve at least one of the following exercises.

Exercise 1

Consider a Hamiltonian system with two degrees of freedom defined in the canonical variables (q_1, q_2, p_1, p_2) by the Hamiltonian

$$H(q_1, q_2, p_1, p_2, \epsilon) = \frac{p_1^2 + p_2^2}{2} + \frac{1}{\epsilon^2}[1 - \cos(\epsilon\omega q_1)\cos(\epsilon q_2)]$$

depending on the positive parameter ϵ. Here ω is an irrational number.

1) Show that the point $q_1 = 0, q_2 = 0$ is a point of stable equilibrium for the Hamiltonian $H(q_1, q_2, p_1, p_2, \epsilon)$.

Hint: verify that the Hamiltonian system describes the motion on the plane of a material point of mass 1 in a potential field. Find the potential of the field.

2) Prove that the new coordinates

$$J_1 = \frac{p_1^2 + \omega^2 q_1^2}{2\omega}, \quad J_2 = \frac{p_2^2 + q_2^2}{2}$$
$$\chi_1 = \arctan\frac{\omega\, q_1}{p_1}, \quad \chi_2 = \arctan\frac{q_2}{p_2}$$

are *action-angle variables* for the Hamiltonian

$$H_0(q_1, q_2, p_1, p_2) = \lim_{\epsilon \longrightarrow 0} H(q_1, q_2, p_1, p_2, \epsilon)$$

i.e., (1) the transformation $(q_1, q_2, p_1, p_2) \mapsto (\chi_1, \chi_2, J_1, J_2)$ is canonical;
(2) the points with the coordinates

$$(\chi_1, \chi_2, J_1, J_2) \quad \text{and} \quad (\chi_1 + 2\pi n_1, \chi_2 + 2\pi n_2, J_1, J_2), \quad \forall\, n_1,\, n_2 \in \mathbb{Z}$$

are identified; (3) the Hamiltonian H_0 in the new coordinates does not depend on χ_1, χ_2:

$$H_0(q_1, q_2, p_1, p_2) = K_0(J_1, J_2)$$

for some function $K_0(J_1, J_2)$.

3) Calculate the generating function

$$W(\chi_1, \chi_2, \tilde{J}_1, \tilde{J}_2, \epsilon) = \tilde{J}_1\chi_1 + \tilde{J}_2\chi_2 + \sum_{k=1}^{2} \epsilon^k F_k(\chi_1, \chi_2, \tilde{J}_1, \tilde{J}_2)$$

of the canonical transformation $(\chi_1, \chi_2, J_1, J_2) \to (\tilde{\chi}_1, \tilde{\chi}_2, \tilde{J}_1, \tilde{J}_2)$ that transforms the Hamiltonian $H(\chi_1, \chi_2, J_1, J_2, \epsilon)$ to the form

$$K(\tilde{\chi}_1, \tilde{\chi}_2, \tilde{J}_1, \tilde{J}_2, \epsilon) = K_0(\tilde{J}_1, \tilde{J}_2) + \epsilon K_1(\tilde{J}_1, \tilde{J}_2) + \epsilon^2 K_2(\tilde{J}_1, \tilde{J}_2) + O(\epsilon^3).$$

4) Find the frequencies of small oscillations near the equilibrium point $q_1 = q_2 = 0$. Determine the ϵ^2-corrections to the frequencies as functions of $\tilde{J}_1$, $\tilde{J}_2$.

Hint: Find the functions $K_0(\tilde{J}_1, \tilde{J}_2)$, $K_1(\tilde{J}_1, \tilde{J}_2)$and $K_2(\tilde{J}_1, \tilde{J}_2)$ and calculate the frequencies of motion associated to the Hamiltonian $H = K_0(\tilde{J}_1, \tilde{J}_2)+ K_1(\tilde{J}_1, \tilde{J}_2) + \epsilon^2 K_1(\tilde{J}_1, \tilde{J}_2)$.

Exercise 2

Given an n-dimensional linear space V over $\mathbb{R}$ and a linear operator

$$A : V \to V$$

such that[1]

$$A^n = 0, \quad \dim \operatorname{Ker} A = 1.$$

1) Prove that all eigenvalues of A are equal to 0.
2) How many linearly independent eigenvectors can have the operator A?
3) Determine the Jordan normal form of the matrix of the operator A.
4) Consider another linear operator

$$B : V \to V$$

such that

$$A\,B - B\,A = A.$$

Prove that the eigenvalues of the operator B form an n-term arithmetic progression.
5) Prove that the linear operator B is diagonalizable, i.e., there exists a basis $f_1, \ldots, f_n$ of the space V consisting of eigenvectors of B:

$$B\,f_i = \lambda_i f_i, \quad i = 1, \ldots, n.$$

Here $\lambda_1, \ldots, \lambda_n$ are some real numbers (the eigenvalues of the operator B).

[1] Recall that the kernel of a linear operator A is defined as the subspace of V consisting of all null-vectors of A:

$$\operatorname{Ker} A := \{x \in V \mid A\,x = 0\}.$$

Exercise 3

Recall that a real *Lie algebra* is a vector space V over $\mathbb{R}$ equipped with a bilinear map (called *Lie bracket*) $[\ ,\ ] : V \times V \to V$ satisfying

$$[v, w] = -[w, v], \quad \forall\, v, w \in V \tag{1}$$

(antisymmetry)

$$[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0, \quad \forall\, u, v, w \in V \tag{2}$$

(Jacobi identity).

i) Let us consider the vector fields (derivations of $C^\infty(\mathbb{R})$) of the form

$$v = a(x)\frac{d}{dx}, \quad a(x) \in C^\infty(\mathbb{R}).$$

Prove that the commutator of derivations defined by the formula

$$[a(x)\frac{d}{dx}, b(x)\frac{d}{dx}]\, f(x) = a(x)\frac{d}{dx}\left(b(x)\frac{d}{dx}f(x)\right) - b(x)\frac{d}{dx}\left(a(x)\frac{d}{dx}f(x)\right) \quad \forall\, f(x) \in C^\infty(\mathbb{R})$$

is again a derivation. Prove that the commutator satisfies the above properties (1), (2) of Lie bracket.

For any nonnegative integer k consider the vector field

$$v_k := x^k \frac{d}{dx}.$$

ii) Determine for which $n \in \mathbb{N}$ the vector fields $\{v_0, v_1, \ldots, v_n\}$ form a basis of a real Lie algebra V_n with respect to the commutator.
iii) For every n found in ii), calculate the center $Z_n := \{v \mid [v, w] = 0, \forall w \in V_n\}$ of the Lie algebra V_n.
iv) Calculate the Killing form on V_n

$$\kappa(v, w) := \mathrm{tr}(\mathrm{ad}_v \cdot \mathrm{ad}_w),$$

where tr denotes the trace and the linear map $\mathrm{ad}_v : V_n \to V_n$ for any $v \in V_n$ is defined by

$$\mathrm{ad}_v\, u := [v, u].$$

v) Is κ nondegenerate? Is κ or $-\kappa$ positive definite? What does it mean for semisimplicity and compactness of the Lie group corresponding to V_n?

Exercise 4

Let us consider a function $f(z)$ of the complex variable z analytic in $\mathbb{C}\backslash(-\infty, 0]$. Suppose that the limitng values

$$f_{\pm}(\lambda) := \lim_{\epsilon\to 0} f(\lambda \pm i\epsilon),$$

exist $\forall\, \lambda \in \mathbb{R}$.

a) Show that the only solution of the following problem:

- $f_+(\lambda) + f_-(\lambda) = 0, \quad \lambda \in (-\infty, 0)$
- $f_+(\lambda) - f_-(\lambda) = 0, \quad \lambda \in (0, +\infty)$
- $f(z) = \sqrt{z} + O(1/\sqrt{z}), \quad |z| \to \infty, \;\; z \in \mathbb{C}\backslash(-\infty, 0]$

is given by

$$f(z) = \sqrt{z},$$

where $\sqrt{z}$ is assumed to be analytic in $\mathbb{C}\backslash(-\infty, 0]$ and positive for $z \in (0, +\infty)$.

b) Let $u \in \mathbb{R}$ is a positive number. Show that the only function $h(z)$ analytic in $\mathbb{C}\backslash(-\infty, u)$ and satisfying

- $h_+(\lambda) = -h_-(\lambda), \quad \lambda \in (-\infty, 0)$
- $h_+(\lambda) = ih_-(\lambda), \quad \lambda \in (0, u)$
- $h_+(\lambda) = h_-(\lambda), \quad \lambda \in (0, +\infty)$
- $h(z) = \sqrt{z} + O(1/\sqrt{z}), \quad |z| \to \infty, \;\; z \in \mathbb{C}\backslash(-\infty, u]$

is given by

$$h(z) = [z(z-u)]^{1/4}, \quad h(\lambda) > 0 \quad \text{for} \quad \lambda \in (u, +\infty).$$

Here as above existence of the limits

$$h_{\pm}(\lambda) := \lim_{\epsilon\to 0} h(\lambda \pm i\,\epsilon) \quad \forall\, \lambda \in \mathbb{R}$$

is assumed.

Exercise 5

1) Check that for any positive real number a the function

$$\psi = e^{-a\,\frac{x^2}{2}}$$

is an eigenfunction of the linear differential operator

$$-\frac{d^2}{dx^2} + a^2 x^2$$

with eigenvalue a.

2) Consider two material points of mass 1 on the line connected by a spring of elastic coefficient 1/2. Recall that the evolution of the wave function $\phi(x, y, t)$ of the system is determined by the Schrödinger equation

$$i\,\frac{\partial \phi}{\partial t} = -\frac{1}{2}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + \frac{1}{4}(x-y)^2\phi$$

(in a suitable system of units). Find the solution to the Schrödinger equation with the initial data

$$\phi(x, y, 0) = e^{-\frac{1}{2}(x^2+y^2)}.$$

3) Prove that

$$\int |\phi(x, y, t)|^2 dx\, dy$$

does not depend on t.

Hint: for the motion of baricenter of the system use Fourier transform.