Scuola Internazionale Superiore di Studi Avanzati, Trieste

Mathematical Physics Sector

Selection for the PhD Courses in Mathematical Physics and in Geometry

Written test, July 14, 2008

Each applicant is required to completely solve at least one of the following exercises. Every answer must be sufficiently motivated.

A. Consider the Hamiltonian of the Kepler problem

$$H(\boldsymbol{r},\boldsymbol{p}) = \frac{1}{2}\mu|\boldsymbol{p}|^2 + \frac{\alpha}{|\boldsymbol{r}|}, \quad \mu > 0, \ \ \alpha < 0,$$

where $\boldsymbol{r} \in M = \mathbb{R}^3\backslash\{0\}$, $(\boldsymbol{r},\boldsymbol{p}) \in T^*M$ and $|\boldsymbol{r}| = \sqrt{r_1^2 + r_2^2 + r_3^2}$, and the corresponding Hamilton equations

$$\frac{d}{dt}r_i = \frac{\partial H}{\partial p_i} = \{r_i, H\},$$
$$\frac{d}{dt}p_i = -\frac{\partial H}{\partial r_i} = \{p_i, H\}, \quad i = 1, \dots, n,$$

where $\{\ ,\ \}$ is the standard Poisson bracket on T^*M defined as

$$\{f, g\} = \frac{\partial f}{\partial r_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial r_i}\frac{\partial f}{\partial p_i}$$

for any $f(\boldsymbol{r},\boldsymbol{p}), g(\boldsymbol{r},\boldsymbol{p}) \in \mathcal{C}^\infty(T^*M)$.

1) Verify that the quantities

$$\boldsymbol{m} = \boldsymbol{r} \times \boldsymbol{p}, \quad \boldsymbol{W} = \boldsymbol{p} \times \boldsymbol{m} + \mu\alpha\frac{\boldsymbol{r}}{|\boldsymbol{r}|}$$

are constants of motions.

2) Using the expression

$$\boldsymbol{W} \cdot \boldsymbol{r},$$

show that $\boldsymbol{r}(t)$ describes a conic section lying in the region

$$\{\boldsymbol{r} \cdot \boldsymbol{m} = 0\} \cap \{\boldsymbol{r} \cdot \boldsymbol{W} > \mu\alpha r\}.$$

Classify the possible conics for different values of $H(\boldsymbol{r},\boldsymbol{p}) = E, \quad E \in \mathbb{R}$. (*Hint:* use the following relations: $\boldsymbol{W} \cdot \boldsymbol{m} = 0$ and $|\boldsymbol{W}|^2 = 2\mu|\boldsymbol{m}|^2E + \mu^2\alpha^2$. Also remind that the polar equation of a conic section with focal parameter f and eccentricity e is given by $r = fe/(1 + e\cos\theta)$).

3) Show that $\boldsymbol{p}(t)$ lies on a circle with centre in $(\boldsymbol{m} \times \boldsymbol{W})/|\boldsymbol{m}|^2$. Compute the radius of the circle.

4) The flows generated by the functions m_i and W_i, $i = 1, 2, 3$ are canonical transformations. Are such transformations point tranformations? (Recall that a canonical transformation $\Phi : T^*M \to T^*M$ is a point transformation if it is induced by a transformation $\phi : M \to M$, so that $\Phi(\boldsymbol{r},\boldsymbol{p}) = (\phi(\boldsymbol{r}), \phi^{*-1}(\boldsymbol{p}))$, $(\phi^*(\boldsymbol{p}))_i = \dfrac{\partial\phi_i(\boldsymbol{q})}{\partial q_j}p_j$).

5) For $H(\boldsymbol{r},\boldsymbol{p}) = E$ verify the commutation relations

$$\{m_i, m_j\} = \epsilon_{ijk}m_k, \ \ \{m_i, W_j\} = \epsilon_{ijk}W_k, \quad \{W_i, W_j\} = -2\mu E\epsilon_{ijk}m_k, \tag{1}$$

for $i, j, k = 1, 2, 3$, where ϵ_{ijk} is non zero only for $i \neq j \neq k$ and it is equal to 1 for an even permutation of indeces and -1 for an odd permutation of indeces.

6) Prove that the Lie algebra (1) generated by the functions m_i and W_i is isomorphic to the following Lie algebras:

(i) for $E < 0$ to $so(4)$, that is, the algebra of 4×4 antisymmetric matrices with real entries;

(ii) for $E > 0$ to the Lie algebra $so(1, 3)$ of the Lorentz group $SO(1, 3)$ defined as

$$SO(1,3) = \{G = ((g_{ij})_{1 \leq i,j \leq 4} \in \mathbb{R} \mid G^t J G = J\}, \quad J = \mathrm{diag}(1, -1, -1, -1);$$

(iii) for $E = 0$ to the Lie algebra e_3 of the group E_3 of isometries of the Euclidean space $\mathbb{R}^3$.

Hint: for $E < 0$ introduce the quantities $A_k^{\pm} = m_k \pm \frac{W_k}{\sqrt{-2\mu E}}$, $k = 1, 2, 3$, and for $E > 0$ introduce $A_k = m_k + i\frac{W_k}{\sqrt{2\mu E}}$, and $\bar{A}_k = m_k - i\frac{W_k}{\sqrt{2\mu E}}$, $k = 1, 2, 3$.

B. Consider a particle in one dimension with potential

$$V(x) = -V_0 \delta\Big(\frac{x}{a}\Big) \tag{2}$$

where V_0 and a are positive real numbers.

(1) Show that the system has a bound state, i.e., a state with negative energy, and compute the corresponding energy.

(2) Study the spectrum of unbound states of the system ($E > 0$), determining the reflection and transmission coefficients for plane waves, defined by

$$\begin{aligned} \psi_k(x) &= \exp(ikx) + R(k)\exp(-ikx) \qquad (x < 0) \\ \psi_k(x) &= T(k)\exp(ikx) \qquad (x > 0) \end{aligned} \tag{3}$$

(3) Show that the conservation of the probability current implies

$$|R(k)|^2 + |T(k)|^2 = 1 \tag{4}$$

(4) Study the analiticity properties (in particular the pole structure) of the functions $R(k), T(k)$ and comment on them in relation with the results of point 1.

(5) Compute the inner product of the plane waves in eq.(3)

$$\int_{-\infty}^{\infty} dx\ \psi_k^*(x)\psi_l(x)$$

[hint: recall that $\int_0^\infty \exp(ikx) = \pi\delta(k) + i\,\mathrm{p.p.}(1/k)$ and use the properties of the principal part].

C. The total electromagnetic force (Lorentz force) due to an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ acting on a particle of charge q and velocity $\mathbf{v}$ in 3-dimensional space is

$$\mathbf{F} = q\mathbf{E} + \frac{q}{c}\mathbf{v} \times \mathbf{B}$$

when $|\mathbf{v}| \ll c$, c being the speed of light.

1. Find the velocity dependent potential U such that the equations of motion can be derived from a Lagrangian of the form $L = T - U$

2. Show that in a constant magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ (when $\mathbf{E} = \mathbf{0}$) the particle moves in a uniform circular motion. Find the radius and the frequency of such a motion. Find a reference system with respect to which the particle is at rest.

3. Do the Lorentz force and the equation of motion get modified in special relativity, when $|\mathbf{v}| \simeq c$? If they are, how? Extend the results obtained at point 2) above to this case and prove that they give back the correct nonrelativistic limit when $|\mathbf{v}| \ll c$.

D. Given a natural number n, denote

$$\mathcal{H}_n := \{X \in Mat(n, \mathbb{C}) \,|\, X^* = X\}$$

the set of $n \times n$ Hermitian matrices. Here

$$X^* := \bar{X}^{\mathrm{T}}$$

is the operation of Hermitian conjugation (i.e., complex conjugation along with matrix transposition).

(1) Prove that $\mathcal{H}_n$ is a linear space over $\mathbb{R}$. Determine the dimension of this space, and construct a basis.

(2) Define a bilinear form on $\mathcal{H}_n$ by the formula

$$< X, Y >= \mathrm{tr}\,(X \cdot Y), \quad X,\, Y \in \mathcal{H}_n \tag{5}$$

where tr stands for the trace of a matrix. Prove that this bilinear form defines a Euclidean structure on $\mathcal{H}_n$.

(3) Denote

$$P_X(\lambda) = \det\,(X - \lambda \cdot \mathrm{Id})$$

the characteristic polynomial of the matrix X. Discuss the properties of the roots of the characteristic polynomial for the case $X \in \mathcal{H}_n$, as well as the properties of the eigenvectors of Hermitian matrices.

(4) For given real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ denote

$$\mathcal{H}_n(\lambda_1, \ldots, \lambda_n) = \{X \in \mathcal{H}_n \,|\, P_X(\lambda_1) = 0,\ P_X(\lambda_2) = 0, \ldots, P_X(\lambda_n) = 0\} \tag{6}$$

the set of all Hermitian matrices with prescribed characteristic roots. Prove that

$$\mathcal{H}_n(\lambda_1, \ldots, \lambda_n) \subset \mathcal{H}_n$$

is a smooth compact submanifold. Determine the dimension of this submanifold.

(5) Consider the particular case $n = 2$. For given real numbers $\lambda_1 < \lambda_2$ compute the area of the submanifold

$$\mathcal{H}_2(\lambda_1, \lambda_2) \subset \mathcal{H}_2$$

with respect to the metric induced on the submanifold by the Euclidean metric of $\mathcal{H}_2$ (see n. 2 above).

E. Let M be the open submanifold of $\mathbb{R}^3$ given by the condition $z > 0$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. Let X, Y be the vector fields on M

$$X = \frac{\partial}{\partial x} + a(y)\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial z} - \frac{1}{z}\frac{\partial}{\partial y}$$

where a is a smooth function. Let Δ be the distribution (subbundle of the tangent bundle) generated by X and Y.

(1) Define the notions of involutivity and integrability of distributions on a differentiable manifold in the sense of Frobenius.

(2) Choose a nonzero function a such that Δ is integrable.

(3) Find a generator of the ideal $\mathcal{I}$ in the algebra of differential forms on M associated with the distribution.

(4) Prove that (with the previous choice of the function a) the ideal $\mathcal{I}$ is a differential ideal, i.e., $d\mathcal{I} \subset \mathcal{I}$.

(5) Find the integral subvarieties of Δ of maximal dimension.

F. Let n be a positive integer, and r another integer with $0 < r < n$. Let $G(r, n)$ be the Grassmann variety parametrizing r-dimensional vector subspaces of a fixed n-dimensional complex vector space.

(1) Recall the definition of $G(r, n)$, and prove that it is an irreducible, smooth algebraic variety. Compute its dimension.

(2) Prove that $G(2, n)$ is isomorphic to the locus of rank 2 matrices in the projective space associated to the vector space of $n \times n$ skew-symmetric matrices.

(3) Let $G = G(2, 4)$; it parametrizes lines in $\mathbb{P}^3$. Let $C \subset \mathbb{P}^3$ be a rational normal curve (i.e., the image of the degree 3 Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$), and let $X \subset G$ be the locus of secant lines to C (i.e., lines that meet C in two points or are tangent to C). Show that X is isomorphic to the projective plane. Find the degree of X in the Plücker embedding of G.

(4) Prove that there is an open subvariety of $G(n, 2n)$ which is isomorphic to $GL(n)$. Hint: use the graph of a map.

G. Let A be a topological space, $B \subset A$ a subspace. Define the quotient of A by B to be the quotient of A by the equivalence relation $a \equiv b$ iff $a = b$ or $a, b \in B$, with the quotient topology.

Let X_n be the quotient of the interval $I = [0, 1]$ by the subset $Y_n = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$.

(1) Find a subspace of $\mathbb{R}^2$ homeomorphic to X_3.

(2) Prove that the fundamental group of X_n is the free group F_n on n generators.

(3) Prove that X_n is not homeomorphic to X_m if $n \neq m$.

(4) Let $\pi : Z \to X_n$ be a topological covering, and assume that Z is connected and π has finite fibres. Prove that Z is homotopy equivalent to X_m, for some m.

(5) Prove that every subgroup G of F_n of finite index is isomorphic to F_m for some m.

(6) Let Σ be a noncompact, noncontractible Riemann surface. Prove that there exists a unique n such that Σ is homotopy equivalent to X_n. Prove that Σ is not homeomorphic to X_n.