Each applicant is required to completely solve at least one of the following exercises. Every answer must be sufficiently motivated.

A. Let us consider $\mathbb{R}^3$ with polar coordinates $(r, \theta, \psi)$. A particle of mass $m$ moves in $\mathbb{R}^3$ on the orbit described by the equation $r = 2a \sin \theta$, $a$ nonzero constant, with velocity $|v| = k/\sin^2 \theta$, $k > 0$.

1. Show that the force is central;
2. determine the force;
3. write the Hamiltonian for the system in polar coordinates and calculate the energy.
4. Study qualitatively, for various values of the energy $E \in \mathbb{R}$ the orbit of a particle in $\mathbb{R}^3$ of mass $m$ attracted towards the origin by a force of intensity $\mu/r^5$ where $\mu > 0$. In particular determine
   • when the orbit is bounded
   • when the orbit is periodic
   • when the orbit has asymptotic points, namely points that are reached in an infinite time.

B. Let $I = [0, L] \subset \mathbb{R}$. In the Hilbert space $\mathcal{H} = L^2(I) \otimes \mathbb{C}^2 \cong L^2(I) \oplus L^2(I)$ consider the Hamiltonian operator

$$
H_\lambda = -\frac{d^2}{dx^2} \otimes I + \lambda \cos x \otimes \sigma_1,
$$

(1)

where

$$
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

and $\frac{d^2}{dx^2}$ is defined through vanishing boundary conditions. Let $t \mapsto \psi(t)$ the solution of the initial value problem

$$
\begin{cases}
   i \frac{d}{dt} \psi(t) = H_\lambda \psi(t), \\
   \psi(0) = \psi_0,
\end{cases}
$$

(2)

Consider first $\lambda = 0$.

1. prove that $H_0$ admits a complete orthonormal system of eigenfunctions;
2. consider $\psi_0 \in \mathcal{H}$ such that $\langle \psi_0, H_0 \psi_0 \rangle < +\infty$. Show that there exist $T > 0$ such that the solution of (2) (for $\lambda = 0$) is $T$-periodic, i.e.

$$
\psi(t + T) = \psi(t), \quad \text{for all } t \in \mathbb{R}.
$$
Consider next $\lambda \neq 0$:

3. prove that every eigenfunction of $H_\lambda$ with a non-degenerate eigenvalue (i.e. corresponding to a one-dimensional eigenspace) is in the form $\begin{pmatrix} \phi \\ \phi \end{pmatrix}$ or $\begin{pmatrix} \phi \\ -\phi \end{pmatrix}$ for a suitable $\phi \in L^2(I)$.

**Hint:** it might be useful to consider the operator $R$ defined by

$$R\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_2 \\ \phi_1 \end{pmatrix}$$

4. discuss whether a solution of (2) with $\langle \psi_0, H_\lambda \psi_0 \rangle < +\infty$ is generically periodic. Provide a sufficient condition (e.g. on the eigenvalues of $H_\lambda$) for periodicity;

5. **optional:** prove that $H_\lambda$ admits a complete orthonormal systems of eigenfunctions;

6. for $\lambda$ sufficiently small, $0 < \lambda < \lambda_0$, discuss how the eigenvalues of $H_\lambda$ depend on $\lambda$ at the first order in perturbation theory.

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C.

I. Let $A : \mathbb{C}^m \to \mathbb{C}^n$ be a linear map of rank $r$ between two complex linear spaces with the standard hermitean structure and let $A^* : \mathbb{C}^n \to \mathbb{C}^m$ be the adjoint map.

1. Prove that the two maps $A^*A : \mathbb{C}^m \to \mathbb{C}^m$ and $AA^* : \mathbb{C}^n \to \mathbb{C}^n$ have the same non zero eigenvalues $\lambda^2_1, \cdots, \lambda^2_r$, and prove that one can chose the eigenvectors $(u_1, \cdots, u_m)$ of $A^*A$ and $(v_1, \cdots, v_n)$ of $AA^*$ to give orthonormal bases for $\mathbb{C}^m$ and $\mathbb{C}^n$.

2. Prove that in the bases above the matrix of $A$ has the form

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \lambda_m \\ 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

when $m < n$, where $\lambda_1, \cdots, \lambda_r$ are the singular values of $A$, i.e. the (positive) square roots of the non zero eigenvalues of $A^*A$ and $AA^*$ and $\lambda_{r+1} = \cdots = \lambda_m = 0$. What happens if $m \geq n$?

3. Let $u$ and $v$ be the unitary matrices whose columns are the eigenvectors above. Prove that in any basis the "singular value decomposition" $A = vD u^*$ holds.

II. Let $G$ be the group of all transformations of $\mathbb{R}$ of the following form

$$x \mapsto ax + b,$$

where $a, b \in \mathbb{R}$, $a \neq 0$.

1. Show that $G$ is a Lie group.

2. Calculate the de Rham cohomology $H^p_{dR}(G)$, $p \in \mathbb{N}$.

3. Find

- the space of left invariant volume 2-forms,
• the space of right invariant volume 2-forms,
• the space of left-and-right invariant volume 2-forms on $G$.

D. A way to introduce the Hopf fibration $\pi: S^3 \to S^2$ is to act with the circle group $S^1$ (complex numbers of unit norm) on the unit sphere $S^3$ in $\mathbb{C}^2$ according to

$$\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2).$$

1. Show that the quotient $S^3/S^1$ is the space $\mathbb{C}P^1$ of complex lines in $\mathbb{C}^2$.
2. Find a diffeomorphism of $\mathbb{C}P^1$ with the 2-dimensional sphere $S^2$.
3. Show that the resulting fibration $\pi: S^3 \to S^2$ is a fibre bundle.
4. For every $u \in S^3$ let $H_u$ be the (unique) complex line in the tangent space $T_u \mathbb{C}^2$ that is contained in the real subspace $T_u S^3 \subset T_u \mathbb{C}^2$. Prove that the assignment $u \mapsto H_u$ defines a connection on the bundle $\pi: S^3 \to S^2$.

E. Let $X$ be an irreducible projective curve in complex $\mathbb{P}^n$ with $n \geq 2$.

1. Define the degree of $X$.
2. Let $p \in \mathbb{P}^n$ be a point, and let $Y$ be the $(n-1)$-dimensional projective space of lines through $p$. Let $\pi: \mathbb{P}^n \setminus p \to Y$ be the morphism associating to a point $q$ the line through $p$ and $q$. Show that $\pi$ does not extend to a morphism $\mathbb{P}^n \to Y$.
3. Assume from now on that $p \in X$ is a regular point. Prove that $\pi$ restricted to $X \setminus p$ extends to a morphism $f: X \to Y$.
4. Let $X' = f(X)$. Give an example of $X$ and $p$ such that $f$ is constant and $X'$ is a point. Give a different example in which $f: X \to X'$ is finite but not birational.
5. Show that if $f: X \to X'$ is birational, then the degree of $X'$ is one less than the degree of $X$.
6. Let $X$ be a smooth, transversal intersection of two smooth quadrics in $\mathbb{P}^3$, and assume that $f$ is birational. Can $X'$ be a nodal curve? A smooth curve? Can it have nonnodal singularities, and if so which ones?