Scuola Internazionale Superiore di Studi Avanzati, Trieste

Mathematical Physics Sector

Selection for the PhD Courses in Mathematical Physics and in Geometry

Written test, July 13, 2009

Each applicant is required to completely solve at least one of the following exercises. Every answer must be sufficiently motivated.

A. Let us consider $\mathbb{R}^3$ with polar coordinates (r, θ, ψ). A particle of mass m moves in $\mathbb{R}^3$ on the orbit described by the equation $r = 2a \sin\theta$, a nonzero constant, with velocity $|\boldsymbol{v}| = k/\sin^2\theta$, $k > 0$.

1. Show that the force is central;
2. determine the force;
3. write the Hamiltonian for the system in polar coordinates and calculate the energy.
4. Study qualitatively, for various values of the energy $E \in \mathbb{R}$ the orbit of a particle in $\mathbb{R}^3$ of mass m attracted towards the origin by a force of intensity μ/r^5 where $\mu > 0$. In particular determine
   - when the orbit is bounded
   - when the orbit is periodic
   - when the orbit has asymptotic points, namely points that are reached in an infinite time.

B. Let $I = [0, L] \subset \mathbb{R}$. In the Hilbert space $\mathcal{H} = L^2(I) \otimes \mathbb{C}^2 \cong L^2(I) \oplus L^2(I)$ consider the Hamiltonian operator

$$H_\lambda = -\frac{d^2}{dx^2} \otimes \mathbb{I} + \lambda \cos x \otimes \sigma_1, \tag{1}$$

where

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $\frac{d^2}{dx^2}$ is defined through vanishing boundary conditions. Let $t \mapsto \psi(t)$ the solution of the initial value problem

$$\begin{cases} i\frac{d}{dt}\psi(t) = H_\lambda \psi(t), \\ \psi(0) = \psi_0, \end{cases} \tag{2}$$

Consider first $\lambda = 0$.

1. prove that H_0 admits a complete orthonormal system of eigenfunctions;

2. consider $\psi_0 \in \mathcal{H}$ such that $\langle \psi_0, H_0 \psi_0 \rangle < +\infty$. Show that there exist $T > 0$ such that the solution of (2) (for $\lambda = 0$) is T-periodic, *i.e.*

$$\psi(t + T) = \psi(t), \qquad \text{for all } t \in \mathbb{R}.$$

Consider next $\lambda \neq 0$:

3. prove that every eigenfunction of H_λ with a non-degenerate eigenvalue (i.e. corresponding to a one-dimensional eigenspace) is in the form $\begin{pmatrix} \phi \\ \phi \end{pmatrix}$ or $\begin{pmatrix} \phi \\ -\phi \end{pmatrix}$ for a suitable $\phi \in L^2(I)$.
Hint: it might be useful to consider the operator R defined by

$$R \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_2 \\ \phi_1 \end{pmatrix};$$

4. discuss whether a solution of (2) with $\langle \psi_0, H_\lambda \psi_0 \rangle < +\infty$ is generically periodic. Provide a sufficient condition (*e.g.* on the eigenvalues of H_λ) for periodicity;

5. optional: prove that H_λ admits a complete orthonormal systems of eigenfunctions;

6. for λ sufficiently small, $0 < \lambda < \lambda_0$, discuss how the eigenvalues of H_λ depend on λ at the first order in perturbation theory.

C.
I. Let $A : \mathbb{C}^m \to \mathbb{C}^n$ be a linear map of rank r between two complex linear spaces with the standard hermitean structure and let $A^* : \mathbb{C}^n \to \mathbb{C}^m$ be the adjoint map.

1. Prove that the two maps $A^*A : \mathbb{C}^m \to \mathbb{C}^m$ and $AA^* : \mathbb{C}^n \to \mathbb{C}^n$ have the same non zero eigenvalues $\lambda_1^2, \cdots, \lambda_r^2$, and prove that one can chose the eigenvectors $(u_1, \cdots u_m)$ of A^*A and $(v_1, \cdots, v_n)$ of AA^* to give orthonormal bases for $\mathbb{C}^m$ and $\mathbb{C}^n$.

2. Prove that in the bases above the matrix of A has the form

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \lambda_m \\ 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

when $m < n$, where $\lambda_1, \cdots, \lambda_r$ are the singular values of A, i.e. the (positive) square roots of the non zero eigenvalues of A^*A and AA^* and $\lambda_{r+1} = \cdots = \lambda_m = 0$. What happens if $m \geq n$?

3. Let $\mathbf{u}$ and $\mathbf{v}$ be the unitary matrices whose columns are the eigenvectors above. Prove that in any basis the "singular value decomposition" $A = \mathbf{v} D \mathbf{u}^*$ holds.

II. Let G be the group of all transformations of $\mathbb{R}$ of the following form

$$x \mapsto ax + b,$$

where $a, b \in \mathbb{R}$, $a \neq 0$.

1. Show that G is a Lie group.
2. Calculate the de Rham cohomology $H^p_{dR}(G)$, $p \in \mathbb{N}$.
3. Find
    - the space of left invariant volume 2-forms,

- the space of right invariant volume 2-forms,
- the space of left-and-right invariant volume 2-forms on G.

D. A way to introduce the Hopf fibration $\pi\colon S^3 \to S^2$ is to act with the circle group S^1 (complex numbers of unit norm) on the unit sphere S^3 in $\mathbb{C}^2$ according to

$$\lambda(z_1, z_2) = (\lambda z_1, \lambda z_2).$$

1. Show that the quotient S^3/S^1 is the space $\mathbb{CP}^1$ of complex lines in $\mathbb{C}^2$.
2. Find a diffeomorphism of $\mathbb{CP}^1$ with the 2-dimensional sphere S^2.
3. Show that the resulting fibration $\pi\colon S^3 \to S^2$ is a fibre bundle.
4. For every $u \in S^3$ let H_u be the (unique) complex line in the tangent space $T_u\mathbb{C}^2$ that is contained in the real subspace $T_uS^3 \subset T_u\mathbb{C}^2$. Prove that the assignment $u \mapsto H_u$ defines a connection on the bundle $\pi\colon S^3 \to S^2$.

E. Let X be an irreducible projective curve in complex $\mathbb{P}^n$ with $n \geq 2$.

1. Define the degree of X.
2. Let $p \in \mathbb{P}^n$ be a point, and let Y be the $(n-1)$-dimensional projective space of lines through p. Let $\pi : \mathbb{P}^n \setminus p \to Y$ be the morphism associating to a point q the line through p and q. Show that π does not extend to a morphism $\mathbb{P}^n \to Y$.
3. Assume from now on that $p \in X$ is a regular point. Prove that π restricted to $X \setminus p$ extends to a morphism $f : X \to Y$.
4. Let $X' = f(X)$. Give an example of X and p such that f is constant and X' is a point. Give a different example in which $f : X \to X'$ is finite but not birational.
5. Show that if $f : X \to X'$ is birational, then the degree of X' is one less than the degree of X.
6. Let X be a smooth, transversal intersection of two smooth quadrics in $\mathbb{P}^3$, and assume that f is birational. Can X' be a nodal curve? A smooth curve? Can it have nonnodal singularities, and if so which ones?