Scuola Internazionale Superiore di Studi Avanzati, Trieste

Mathematical Physics Sector

Selection for the PhD Courses in Mathematical Physics and in Geometry

Written test, July 20, 2010

Each applicant is required to completely solve at least one of the following exercises. Every answer must be sufficiently motivated.

A. Consider a particle in one dimension with potential

$$V(x) = \frac{1}{2} m\omega^2 x^2 \mathbf{1} - gx\sigma_{\mathbf{3}}$$

where m, ω, g are positive real numbers, $\mathbf{1}$ is the identity matrix and $\sigma_{\mathbf{3}} = \mathrm{diag}(1, -1)$.

1. calculate the spectrum of the system and the ground state wave-function.
2. find the modification to the ground state energy in presence of a perturbation term $H_1 = \epsilon\sigma_{\mathbf{1}}$, with $\sigma_{\mathbf{1}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, at first order in perturbation theory.
3. discuss the case $\omega = 0, \epsilon = 0$ and find an integral representation for the one particle wave-function in this case. Compare the qualitative behaviour of this solution with the one of case 1.

B.

1. Prove that the set of functions of the form

$$\psi(x,t) = c(t)^{-1} e^{\frac{-(x-q(t))^2}{2c(t)^2}} e^{ip(t)x} \qquad c(t),\ x(t),;p(t) \in R$$

is invariant (as set) under the flow defined by the Schroedinger equation

$$i\frac{\partial\phi}{\partial t} = -\frac{1}{2}\frac{\partial^2\phi}{\partial x^2}$$

and find the orbit $\{q(t), p(t)\}$ corresponding to the initial datum $\{q_0,\ p_0,\ c_0\}$.

2. Show that irrespective of $\{q_0,\ p_0\, c_0\}$ one has for all $R > 0$

$$lim_{t\to\infty} \int_{|x|<R} |\psi(x,t)|^2 dx = 0$$

3. Show that for the Schroedinger equation

$$i\frac{\partial\phi}{\partial t} = -\frac{1}{2}\frac{\partial^2\phi}{\partial x^2} + \frac{1}{2}x^2\phi - \frac{1}{2}\phi$$

the same invariance holds but irrespective of $\{q_0,\ p_0,\ c_0\}$ one has

$$lim_{R\to\infty} sup_t \int_{|x|>R} |\psi(x,t)|^2 dx = 0$$

C. Consider in $R^1 \times R^1$ the system with Hamiltonian

$$H(q,p) = \frac{1}{2}|p|^2 - \frac{1}{2}|q|^2 + \frac{1}{4}(|q|^2)^2.$$

1. Perform its qualitative analysis (equilibrium points, their stability, periodic orbits, asymptotic motions, ..).
2. Study the behavior of the periods when the energy of the orbit tends to infinity, and when it tends to zero from above and from below.
3. Study the same system but now in $R^2 \times R^2$ finding periodic orbits, invariant tori, omoclinics.
4. Find action-angle variables for the system.

D. Consider the motion of a point of mass $m = 1$ that is moving in the three-dimensional space under the action of a central potential

$$V(\boldsymbol{r}) = -\frac{1}{2|\boldsymbol{r}|^2}, \quad \boldsymbol{r} = (r_1, r_2, r_3)$$

1. Write the Hamiltonian of the system.
2. Show that the angular momentum $\boldsymbol{L} = \boldsymbol{r} \wedge \dot{\boldsymbol{r}}$ is a constant of motion.
3. Discuss qualitatively the motion for different values of $|\boldsymbol{L}|$ and the energy E.
4. Determine in which cases there are periodic orbits.
5. When $|\boldsymbol{L}| = 1$, integrate the equations of motion.
6. When $|\boldsymbol{L}| < 1$, and $E < 0$ integrate the equations of motion. When the particle is at $t = 0$ at distance $\boldsymbol{r}_0$ from the origin, determine the time needed to reach the origin.

E. Let n be a positive integer, ω a complex number such that

$$\omega^n = 1, \quad \omega \neq 1.$$

Compute the sum

$$S_n(\omega) = 1 + 2\omega + 3\omega^2 + \cdots + n\omega^{n-1}.$$

F. Let V and W be vector spaces of dimension 3 over the complex numbers. Let Z be the projectivization (i.e., the variety of one-dimensional subspaces) of the space of linear maps from V to W. Inside Z. let X be the locus of maps of rank one and Y that of maps of rank ≤ 2. Let G be the group $GL(V) \times GL(W)$ acting on Z by composition on either side.

1. show that X is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^2$;
2. show that Y is an irreducible rational variety, and that its singular locus is X;
3. describe the orbits of the action of G on Z;
4. discuss how the previous statements must be modified if we assume that V and W have dimension m and n respectively, with m and n integers ≥ 2.

G. Consider the set $SO(3)$ of all 3×3 orthogonal real matrices with unit determinant as a subset of $\mathbb{R}^9$ and use this to show that it is manifold. Hence prove that it is a Lie group.

Moreover show that with this manifold structure, $SO(3)$ is homeomorphic to the real projective 3-space $\mathbb{RP}^3$.

Compute the fundamental group of $SO(3)$, and show that the universal covering space of $SO(3)$ is $SU(2)$.

H. Denoting by $r = \sqrt{x^2 + y^2 + z^2}$ the radial coordinate in $\mathbb{R}^3 \backslash \{0\}$, consider the vector field X of components x/r^3, y/r^3, z/r^3.

1. Check that X is a gradient and compute its divergence.
2. Compute the flux of X through the unit sphere, with normal vector outwardly oriented.
3. Let ω be the 2-form defined by

$$\omega(Y, Z) = < X, Y \wedge Z >$$

   where $<,>$ is the standard inner product in $\mathbb{R}^3$ and $\wedge$ is the vector product in $\mathbb{R}^3$. Prove that ω is closed.

4. Is ω exact?