Scuola Internazionale Superiore di Studi Avanzati, Trieste

Mathematical Physics Sector

Selection for the PhD Courses in Mathematical Physics and in Geometry

Written test, 12 September, 2011

Each applicant is required to completely solve at least one of the following exercises. Every answer must be sufficiently motivated.

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1. Consider n harmonic oscillators with Hamiltonian

$$H(\boldsymbol{p},\boldsymbol{q}) = \sum_{i=1}^{n} \frac{p_i^2 + m_i^2\omega_i^2 q_i^2}{2m_i},$$

with $\boldsymbol{p} = (p_1,\ldots,p_n) \in \mathbb{R}^n$ and $\boldsymbol{q} = (q_1,\ldots,q_n) \in \mathbb{R}^n$ and m_i and ω_i real positive numbers for $i = 1,\ldots,n$.

a) Show that the Hamiltonian system has n independent conserved quantities in involution with respect to the standard Poisson brackets

$$\{q_i, q_j\} = 0, \qquad \{q_i, p_j\} = \delta_{ij}, \qquad \{p_i, p_j\} = 0.$$

b) Show that the change of coordinates $(p_i, q_i) \to (J_i, \phi_i)$ where

$$p_i = \sqrt{2m_i\omega_i J_i}\cos\phi_i, \quad q_i = \sqrt{\frac{2J_i}{m_i\omega_i}}\sin\phi_i,$$

is a canonical transformation and determine the domain of the transformation and the generating function.

c) Show that the coordinates $(\boldsymbol{J},\boldsymbol{\phi})$, $\boldsymbol{J} = (J_1.\ldots,J_n)$ and $\boldsymbol{\phi} = (\phi_1,\ldots,\phi_n)$ are action-angle variables for the Hamiltonian $H(\boldsymbol{p},\boldsymbol{q})$ and write the Hamiltonian in such coordinates. Describe the common p level surface M of the first integrals where the motion takes place.

d) In the case $n = 3$ suppose that the frequencies ω_1, ω_2 and ω_3 satisfy the relation

$$\omega_1 + 2\omega_2 - 4\omega_3 = 0, \quad \omega_1 - \omega_2 = 0.$$

Furthermore, consider the change of coordinates

$$\tilde{J} = (M^T)^{-1}J, \quad \tilde{\phi} = M\phi$$

where

$$M = \begin{pmatrix} 1 & 2 & -4 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

and show that it is a canonical transformation. Show that the transformed Hamiltonian assumes the form

$$\tilde{K}(\tilde{J}) = (\omega_3 - \omega_2)\tilde{J}_3,$$

and discuss the result.

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2. Given a differentiable manifold X, let $\Omega_c^k(X)$ be the space of differential k-forms on X with compact support.

a) Prove that if $\omega \in \Omega_c^k(X)$ then $d\omega \in \Omega_c^{k+1}(X)$. It is therefore possible to define a cohomological theory $H_{dR,c}^{\bullet}(X)$ — the de Rham cohomology of X with compact supports.

b) Compute the de Rham cohomology with compact supports of $\mathbb{R}$.

c) Let X be a compact oriented differentiable manifold of dimension n, and let $\xi \in H_{dR}^n(X)$. Prove that there exists an open subset $U \subset x$ diffeomorphic to the open n-ball, and a closed n-form ω such that $\omega_{|U} = 0$ and $[\omega] = \xi$.

d) Prove that $H_{dR,c}^n(\mathbb{R}^n) \simeq H_{dR}^n(S^n)$.

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3. (a) Let x be a point in the affine space $\mathbb{A}^1$. State and prove a criterion for which a polynomial $p(t) \in k[t]$ is a local parameter for $\mathbb{A}^1$ in x.

(b) Let O_x be the local ring of a point x in affine space $\mathbb{A}^n$, with $n > 1$, and let $f_1, \ldots, f_n$ be germs in O_x. Let Y_i be the hypersurface given around x by $f_i = 0$, with $i = 1, \ldots, n$.

(c) Explain what it means that the hypersufaces Y_i meet transversally at x.

(d) Assuming that the hypersufaces Y_i meet transversally at x, show that $(f_1, \ldots, f_n)$ is system of local parameters around x.

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4. (a) For an arbitrary positive integer k prove existence of a polynomial $P_k(x)$ of degree k

$$P_k(x) = a_{k,0}x^k + a_{k,1}x^{k-1} + \cdots + a_{k,k}$$

such that

$$\cos k\phi = P_k(\cos\phi).$$

(b) Prove that all coefficients of the polynomials $P_k(x)$ are integers. Compute the leading coefficient $a_{k,0}$ of these polynomials.

(c) For a given positive integer n and arbitrary numbers $\phi_1, \ldots, \phi_n$ compute the determinant D_n of the following $n \times n$ matrix

$$D_n = \det \begin{pmatrix} 1 & \cos\phi_1 & \cos 2\phi_1 & \ldots & \cos(n-1)\phi_1 \\ 1 & \cos\phi_2 & \cos 2\phi_2 & \ldots & \cos(n-1)\phi_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \cos\phi_n & \cos 2\phi_n & \ldots & \cos(n-1)\phi_n \end{pmatrix}.$$

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5. Consider a quantum mechanical system whose Hilbert space of states is $\mathbb{C}^2$, and has Hamiltonian

$$\begin{pmatrix} E_0 e^{t/\omega_0} & E_1 \\ E_1 & E_0 e^{t/\omega_0} \end{pmatrix}$$

(a) Describe the time evolution of the system.

(b) Supposing that at $t = 0$ the system is in the state $\Psi(t)|_{t=0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, find the value t_{max} of the time at which the transition probability to the state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is maximal, and the value of that probability.

(c) discuss the validity of the Ehrenfest theorem $\frac{d}{dt}\langle\mathcal{O}\rangle = \langle\frac{\partial}{\partial t}\mathcal{O}\rangle + \frac{1}{i\hbar}\langle[\mathcal{O}, H]\rangle$ for a generic observable $\mathcal{O}$ and present detailed calculations in the limit $\omega_0 \to \infty$ for the observable $\mathcal{O} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.