Scuola Internazionale Superiore di Studi Avanzati, Trieste

Area of Mathematics

Selection for the PhD Courses in Mathematical Physics and in Geometry

Written test, September 10, 2012

Each applicant is required to completely solve at least one of the following exercises. Every answer must be sufficiently motivated.

A.

Consider the matrix

$$H := \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + q\ \sigma_z \otimes \sigma_z,$$

where $q \in \mathbb{C}$ is a parameter and

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1. For which $\alpha \in \mathbb{R}$ is H invariant under the adjoint action of $U \otimes U$, where $U = \exp(i\alpha\sigma_z)$?
2. For which values of q is H Hermitian and for which is positive?
3. Compute $Z(\beta) := \mathrm{Tr}\exp(\beta \mathrm{H})$, for $\beta \in \mathbb{R}$.
4. Find the eigenvectors of $\exp(\beta H)$.

B.

1. State the Mayer-Vietoris theorem for computing the de Rham cohomology of a differentiable manifold that can be written as the union of two open subsets.

2. Compute the de Rham cohomology of the following spaces.

2a. A 2-dimensional torus $T = S^1 \times S^1$.

2b. T minus a point.

2c. A compact oriented surface with 2 holes.

3. Use induction to compute the de Rham cohomology of a compact oriented surface with n holes.

C.

Consider the family of affine algebraic plane curves over $\mathbb{C}$ given by the equation

$$ax^3 + x^2 - y^2 = 0,$$

where a is a complex parameter. Discuss the following questions for varying a.

1. Is the curve irreducible?
2. Find the singular points of the curve. Which type of singularities are displayed?
3. Write the equation of the total transform of the curve.
4. Find the normalization of the curve.

D.

Consider a point particle of mass m and Cartesian coordinates $\vec{r} = (x, y, z) \in \mathbb{R}^3$, with central potential energy $V = V(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$. The function $V(r)$ is smooth for any $r > 0$. The force acting on the particle is $\vec{F} = -\vec{\nabla}V(r)$, where $\vec{\nabla}V(r)$ is the gradient of the function $V(r)$.

1. Prove that the orbit of the particle either lies in the invariant plane orthogonal to the angular momentum $\vec{L} := \vec{r} \times m\dot{\vec{r}}$ when $\vec{L} \neq 0$, or on a line through the origin $(0, 0, 0)$ when $\vec{L} = 0$. (Here $\times$ is the vector (cross) product and $\dot{\vec{r}}$ is the time derivative of $\vec{r} = \vec{r}(t)$).

In the following, assume that $\vec{L} \neq 0$.

2. Let M be the invariant plane orthogonal to $\vec{L}$. For simplicity, assume that $\vec{L} = (0, 0, L)$, $L \in \mathbb{R}\backslash\{0\}$, and introduce polar coordinates (r, θ) on M, defined by $x = r\cos\theta$, $y = r\sin\theta$, $r > 0$, $\theta \in [0, 2\pi)$. Write the Lagrangian $\mathcal{L}(r, \theta, \dot{r}, \dot{\theta})$ of the system on the tangent bundle TM. Prove that there exist two constants of motion.

3. Write the Hamiltonian $H(r, \theta, p_r, p_\theta)$ of the system on T^*M. Show, in the Hamiltonian formalism, that the Hamiltonian and the momentum p_θ are two constants of motion. Show that the system is integrable according to Liouville.

4. An orbit in T^*M lies on an invariant surface defined by $H(r, \theta, p_r, p_\theta) = E$ and $p_\theta = L$, where E, $L \in \mathbb{R}$ are constants. Let $V(r) = kr^\alpha$, where k and α are real numbers such that $k \neq 0$. Find the conditions on k, α, E and L such that there exist canonical action-angle variables (namely, the orbit lies on some compact Arnold torus).

5. Compute *explicitly* the action variables when $\alpha = 2$ and $k > 0$. Write the Hamiltonian as a function of the action variables and say if the motion is periodic. [*Hint:*

$$\int_{\xi_1}^{\xi_2} \sqrt{1 - \frac{A}{\xi^2} - B\xi^2}\, d\xi = \pi \left(\frac{1}{4\sqrt{B}} - \frac{\sqrt{A}}{2} \right),$$

where $A, B > 0$, $4AB \leq 1$ *and* $0 < \xi_1 < \xi_2$ *are the two positive roots of* $1 - \frac{A}{\xi^2} - B\xi^2 = 0$.]

E.

On Minkowski spacetime $\mathbb{R} \times \mathbb{R}^3$, the second set of the vacuum Maxwell equations

$$\nabla \cdot E = \rho, \qquad \nabla \times B - \partial E/\partial t = j \tag{1a}$$

$$\nabla \cdot B = 0, \qquad \nabla \times E + \partial B/\partial t = 0 \tag{1b}$$

implies the existence of potentials (A, Φ), defined up to "gauge transformations"

$$A \to A' = A + \nabla\Lambda, \qquad \Phi \to \Phi' = \Phi - \partial\Lambda/\partial t,$$

such that

$$E = -\nabla\Phi - \partial A/\partial t, \qquad B = \nabla \times A.$$

1. Show that one can choose a set of potentials satisfying the Lorentz gauge condition

$$\nabla \cdot A + \partial\Phi/\partial t = 0 \tag{2}$$

in such a way that the equations (1) are equivalent to the wave equations

$$\nabla^2\Phi - \partial^2\Phi/\partial t^2 = -\rho$$

$$\nabla^2 A - \partial^2 A/\partial t^2 = -j \tag{3}$$

2. Show that the Lorentz condition (2) is preserved under the family of restricted gauge transformations such that $\nabla^2\Lambda - \partial^2\Lambda/\partial t = 0$.

3. Show that the equation (2) and (3) are covariant under the Poincaré group of spacetime transformations

$$X' = M\,X + b$$

with $M^t\eta M = \eta$. ($X = (t, x, y, z)$ and $\eta = \mathrm{diag}(-1, 1, 1, 1)$). What are the transformations of the potentials (Φ, A) and the sources (ρ, j)?

4. Find the conditions on E, B such that there exists a Poincaré transformation after which $E' = 0$ (or $B' = 0$). Show in particular that a purely electrostatic field cannot be transformed into a purely magnetostatic field.

F.

Consider the quantum dynamics of a particle of mass m in the real plane $\mathbb{R}^2$ with coordinates $\mathbf{q} = (q_1, q_2)$ and classical Lagrangian

$$L = \frac{m}{2} \sum_{i=1,2} \dot{q}_i \dot{q}_i + \frac{B}{2} \sum_{i,j=1,2} \epsilon^{ij} q_i \dot{q}_j - \frac{k}{2} \sum_{i=1,2} q_i q_i, \tag{1}$$

where B and k are two real positive constants and ϵ^{ij} the totally antisymmetric tensor in two dimensions with component $\epsilon^{12} = 1$.

1. Calculate the Hamiltonian of the system and quantize the system by imposing the canonical commutation relations. Solve the Heisenberg equations of the quantum system.

2. Calculate the simultaneous eigenstates of the Hamiltonian and of the angular momentum operator $M = \sum_{i,j=1,2} \epsilon^{ij} q_i p_j$ and their spectrum of eigenvalues.

[**Hint**: write the total Hamiltonian as $H = H_{red} - \frac{B}{2m} M$. Note that eigenfunctions of M with eigenvalue $+\lambda$ and $-\lambda$ are degenerate in the spectrum of H_{red} and use this fact to write the eigenfunctions of H in the form $\Psi(\mathbf{q}) = \chi(\mathbf{q})\psi(\mathbf{q})$ were $\chi(\mathbf{q})$ are eigenfunctions of M and $\psi(\mathbf{q})$ eigenfunctions of H_{red} with zero eigenvalue w.r.t. M]

3*. Consider the Lagrangian L_0 which is obtained from (1) by setting $m = 0$. Repeat the calculations of points (1) and (2) for the Lagrangian L_0.

4*. Discuss if and how the solution of the Heisenberg equations and the spectrum of eigenvalues of the above two systems, described respectively by H and H_0, are related in the $m \to 0$ limit.

[**Hint**: study point (2) using polar coordinates. Use separation of variables between the angular and radial part, and write for the radial part the ansatz $R(r) = r^{|\lambda|} \exp(-\alpha r^2/2) u(\alpha r)$. Expand $u(\alpha r)$ in power series and impose that its behavior for $r \to 0$ and $r \to \infty$ does not affect the asymptotic behavior of $R(r)$ in these limits. Recall that the Laplacian in angular coordinates is $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right)$].