Scuola Internazionale Superiore di Studi Avanzati, Trieste

Area of Mathematics

Selection for the PhD Courses in Mathematical Physics and in Geometry

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Each applicant is required to solve at least one of the following exercises. Every answer must be sufficiently motivated.

A.

Denote the Cartesian coordinates of a point $\mathbf{x} = (x^1, x^2, x^3, x^4, x^5, x^6) \in \mathbb{R}^6$ with the notation $(a_1, a_2, a_3, l_1, l_2, l_3)$.
In $\mathbb{R}^6$ consider a *skew-symmetric* tensor $\sum_{\mu<\nu} \pi^{\mu\nu} \frac{\partial}{\partial x^\mu} \wedge \frac{\partial}{\partial x^\nu}$ of type $(2,0)$ with components $\pi^{\mu\nu}(\mathbf{x}) := \{x^\mu, x^\nu\}$, $\mu, \nu = 1,..,6$, where $\{x^\mu, x^\nu\}$ in the above notations is defined as follows:

$$\{a_i, a_j\} := 0, \quad \{a_i, l_j\} := -\sum_{k=1}^{3} \epsilon_{ijk} a_k, \quad \{l_i, l_j\} := -\sum_{k=1}^{3} \epsilon_{ijk} l_k, \qquad i, j \in \{1,2,3\},$$

$$\epsilon_{ijk} = \begin{cases} 1 & (i,j,k) \text{ even permutation of } (1,2,3) \\ -1 & (i,j,k) \text{ odd permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

Answer the following questions:

1) Prove that $(\mathbb{R}^6, \{\cdot,\cdot\})$ is a Poisson manifold with Poisson brackets

$$\{f, g\} := \sum_{\mu,\nu=1}^{3} \{x^\mu, x^\nu\} \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial x^\nu}, \qquad f, g \in \mathcal{C}^\infty(\mathbb{R}^6).$$

Let I_1, I_2, I_3 be positive constants and consider the following Hamiltonian:

$$H := \frac{l_1^2}{2I_1} + \frac{l_2^2}{2I_2} + \frac{l_3^2}{2I_3} - (a_1 + a_2 + a_3), \tag{1}$$

Write the Hamilton equations $\frac{dx^\mu}{dt} = {X_H}^\mu(x)$, $\mu = 1, ..., 6$, where X_H is the Hamiltonian vector field of H with respect to the brackets $\{\cdot,\cdot\}$.

2) Give the definition of Casimir functions, symplectic leaves and symplectic foliation. Prove that:

2.i) For $(a_1, a_2, a_3) \neq 0$, $(l_1, l_2, l_3) \neq 0$, there are two (and only two) independent Casimir functions $\Phi_1(a_1, a_2, a_3)$ and $\Phi_2(a_1, a_2, a_3, l_1, l_2, l_3)$, and compute them explicitly.

2.ii) On the subspace $(a_1, a_2, a_3) = 0$, there is only one independent Casimir function $\Phi_3(l_1, l_2, l_3)$, and compute it explicitly.

3) Consider the case $(a_1, a_2, a_3) = 0$, with $\Phi_3 = l_1^2 + l_2^2 + l_3^2$.

3.i) Describe geometrically the orbits of the Hamiltonian (1) when $I_1 < I_2 < I_3$.

3.ii) Let $L \geq 0$ be a constant. Introduce the coordinates ϑ, φ on the symplectic leaf $\Phi_3(l_1, l_2, l_3) = L^2$, as follows

$$l_1 = L\ \sin\vartheta\ \cos\varphi, \quad l_2 = L\ \sin\vartheta\ \sin\varphi, \quad l_3 = L\ \cos\vartheta; \qquad \vartheta \in [0, \pi],\ \varphi \in [0, 2\pi).$$

Write $\{\vartheta, \varphi\}$ on the symplectic leaf and the Hamilton equations for ϑ, φ.

3.iii) Show that the above Hamilton equations are integrable by means of one quadrature.

4) Consider the case $(a_1, a_2, a_3) \neq 0$. Let $a > 0$ and $c \in \mathbb{R}$ be the constants which identify a symplectic leaf $\Phi_1 = a^2$, $\Phi_2 = c$. The change of coordinates $(a_1, a_2, a_3, l_1, l_2, l_3) \mapsto (a, \theta, \phi, P_a, P_\theta, P_\phi)$ defined by

$$a_1 = a\sin\theta\cos\phi, \quad a_2 = a\sin\theta\sin\phi, \quad a_3 = a\cos\theta,$$

$$l_1 = \sin\phi\ P_\theta + \cos\phi\frac{\cos\theta}{\sin\theta}\ P_\phi - \frac{\cos\phi}{\sin\theta}\ P_a\ + \frac{c}{a^2}\ a_1$$

$$l_2 = -\cos\phi\ P_\theta + \sin\phi\frac{\cos\theta}{\sin\theta}\ P_\phi + \frac{\sin\phi}{\sin\theta}\ P_a\ + \frac{c}{a^2}\ a_2, \qquad l_3 = -P_\phi\ + \frac{c}{a^2}\ a_3.$$

reduces the Poisson brackets to the form: $\{a, \cdot\} = \{P_a, \cdot\} = 0$, and

$$\{\theta, \phi\} = 0, \quad \{\theta, P_\phi\} = \{\phi, P_\theta\} = 0, \quad \{\theta, P_\theta\} = \{\phi, P_\phi\} = 1, \quad \{P_\theta, P_\phi\} = \frac{c}{a}\sin\theta$$

[*Note: it is not required to verify the above expressions.*] The above brackets imply that $\theta, \phi, P_\theta, P_\phi$ are coordinates on a symplectic leaf. Why?

5) Restrict to a symplectic leaf as in point 4). Show that there exist smooth functions $f_1(\theta, \phi)$ and $f_2(\theta, \phi)$ such that the *local* change of coordinates $q^1 = \theta, q^2 = \phi$, $p_1 = P_\theta + f_1(\theta, \phi)$, $p_2 = P_\phi + f_2(\theta, \phi)$, reduce $\{\cdot, \cdot\}$ to the canonical form $\{q^i, q^j\} = \{p_i, p_j\} = 0$, $\{q^i, p_j\} = \delta^i_j$, $i, j \in \{1, 2\}$. Prove that $\{\cdot, \cdot\}$ cannot be *globally* reduced to the canonical form by the change of coordinates above, except for the case $c = 0$.

B.

1) Fix $L > 0$. Prove that the operator $-\frac{d^2}{dx^2}$ acting on the functions in $\mathcal{C}^2(0, L)$ vanishing in a neighborhood of 0 and L, is neither closed nor self-adjoint in $L^2(0, L)$.

2) Prove that the same operator, acting on the functions in $\mathcal{C}^2(0,L)$ which vanish at $x=0$ and $x=L$, is closable and its closure is self-adjoint. Denote this self-adjoint operator by H_L.

3) Find the spectrum of the self-adjoint operator H_L defined at point (2).

4) Determine (as a series) the solution $t \mapsto \psi(t)$ to the Schrödinger equation $\mathrm{i}\frac{d}{dt}\psi = H_L\psi$, considering as initial datum the characteristic function of the interval $[0, \frac{L}{2}]$. Prove that this solution $\psi(t)$ is periodic and compute explicitly the period.

5) For $V \in L^\infty(0,1)$, let $E_n(\lambda)$ be the n-th eigenvalue of the operator $H(\lambda) := H_1 + \lambda V$ in $L^2(0,1)$. Prove that

$$|E_n(1) - E_n(0)| \le \|V\|_\infty.$$

6) Fix $L>0$ and $M>0$, with $L \neq M$. In the Hilbert space $\mathcal{H} = L^2(0,L) \otimes L^2(0,M)$, consider the solution $t \mapsto \Psi(t)$ to the Schrödinger equation corresponding to the Hamiltonian operator

$$H_{L,M} = H_L \otimes 1 + 1 \otimes H_M.$$

Assume that the initial datum Ψ_0 is contained a subspace of $\mathcal{H}$ spanned by a finite number of eigenfunctions of $H_{L,M}$. Find a condition on L and M which implies that $t \mapsto \Psi(t)$ is periodic for every Ψ_0 (possibly with a Ψ_0-dependent period).

C.

1) Let $\xi \in \mathbb{C}$. Consider the map:

$$z = \frac{1}{2}\left(\xi + \frac{1}{\xi}\right).$$

- Show that the map is conformal on the domain $|\xi| \ge 1$.
- Determine the image of the domain $|\xi| > 1$ and $|\xi| = 1$.
- Determine the inverse map.

2) Show that if $u(z)$ is a harmonic function in a domain $\mathcal{D}$ of the complex plane then

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta,$$

where the disk centred at z and radius r is contained in $\mathcal{D}$.

3) Now consider the function

$$u(z) = \frac{1}{\pi}\int_{-1}^{1} \log\frac{1}{|z-t|}\frac{dt}{\sqrt{1-t^2}},$$

where $z \in \mathbb{C}$ and $|z-t|$ denotes the modulus.

- Show that $u(z)$ is a harmonic function for $|z| > 1$.
- Show that

$$u(z) = \begin{cases} \log 2, & z \in [-1,1] \\ \log 2 - \log|z + \sqrt{z^2-1}|, & z \in \mathbb{C}\backslash[-1,1]. \end{cases}$$

Hint: use the substitution $t = \cos\theta$ *and* $z = \xi + \frac{1}{\xi}$ *and show that*

$$|z - \cos\theta| = \frac{1}{2}|\xi - s||\xi^{-1} - s|, \quad s = e^{i\theta}.$$

D.

In the following $H^\bullet(X)$ will denote the de Rham cohomology of an n-dimensional differentiable manifold X. Let $\Omega_c^k(X)$ be the space of differential k-forms on X with compact support.

a) Prove that $d\omega \in \Omega_c^{k+1}(X)$ if $\omega \in \Omega_c^k(X)$.

As a consequence, $(\Omega_c^\bullet(X), d)$ is a differential complex. Let $H_c^\bullet(X)$ be its cohomology (the de Rham cohomology of X with compact supports).

b) Compute the groups $H_c^k(\mathbb{R})$ from the definition.

c) Is the de Rham cohomology with compact supports a homotopy invariant?

d) Discuss and prove, if possible, the following statement: if X is orientabile, the pairing

$$\Omega^k(X) \times \Omega_c^{n-k}(X) \to \mathbb{R}, \qquad (\eta, \omega) \mapsto \int_X \eta \wedge \omega$$

induces an isomorphism $H^k(X)^* \simeq H_c^{n-k}(X)$ (Poincaré duality).

e) Compute the groups $H_c^k(\mathbb{R}^n)$ for all $n \geq 1$.

If $U \subset V$ are open subsets of X there is a natural inclusion $j_{U,V}: \Omega^k_c(U) \hookrightarrow \Omega^k_c(V)$ (explain).

f) Let $X = U \cup V$, with U, V open and connected. Prove that for all k the sequence

$$0 \to \Omega^k_c(U \cap V) \xrightarrow{a} \Omega^k_c(U) \oplus \Omega^k_c(V) \xrightarrow{b} \Omega^k_c(X) \to 0$$

is exact. The morphisms a and b are defined as

$$a(\omega) = (j_{U\cap V,U}(\omega), -j_{U\cap V,V}(\omega)), \qquad b(\eta,\tau) = j_{U,X}(\eta) + j_{V,X}(\tau).$$

g) From the exactness of this sequence deduce a long exact sequence for the de Rham cohomology groups with compact support.

h) Write the latter sequence for the case $X = S^1 \times \mathbb{R}$ and check that via the isomorphism in item d) it is compatible with the Mayer-Vietoris sequence for the usual de Rham cohomology.

E.

Let $C = \{(x,y) \in \mathbb{A}^2_{\mathbb{C}} \,|\, f(x,y) = 0\}$ be the plane algebraic curve defined by the polynomial $f \in \mathbb{C}[x,y]$ and let $L = \{(a,b) + \lambda(u,v) \,|\, \lambda \in \mathbb{C}\}$ be a line passing through $P = (a,b) \in \mathbb{A}^2_{\mathbb{C}}$ not contained in C. The intersection multiplicity between C and L in P, $\mathrm{mult}_P(C \cap L)$, is defined as the multiplicity of the zero of $f(a+\lambda u, b+\lambda v)$ in $\lambda = 0$.

a) Let C and L as above. Prove that the following inequality holds true:

$$\sum_{P\in L} \mathrm{mult}_P(C \cap L) \le d\,,$$

where $d = \deg(f)$ is the degree of f, i.e. the degree of C.

b) Let $C \subset \mathbb{A}^2_{\mathbb{C}}$ be a plane algebraic curve as above. For any $P = (a,b) \in \mathbb{A}^2_{\mathbb{C}}$ the multiplicity of C in P is defined as follows:

$$\mathrm{mult}_P(C) = \min\{\mathrm{mult}_P(C \cap L) \,|\, L \subset \mathbb{A}^2_{\mathbb{C}} \text{ line through } P\}\,.$$

i) Prove that there is an expression as follows:

$$f(x,y) = f_\mu(x-a,y-b) + f_{\mu+1}(x-a,y-b) + \ldots + f_d(x-a,y-b)\,,$$

where, for any $\ell = \mu,\ldots,d$, f_ℓ is a homogeneous polynomial of degree $\ell \ge 0$ and $f_\mu \not\equiv 0$. Conclude that $\mathrm{mult}_P(C) = \mu$.
(Hint: after an appropriate translation, one can assume that $P = (0,0)$.)

ii) Let $f \in \mathbb{C}[x,y]$ be an irreducible polynomial of degree d. Assume that $P = (0,0) \in C = \{f = 0\}$ is of multiplicity $\mathrm{mult}_P(C) = d-1$. Prove that the map

$$\varphi\colon \mathbb{A}^1_{\mathbb{C}} \dashrightarrow \mathbb{A}^2_{\mathbb{C}}$$
$$u \mapsto \left(-\frac{f_{d-1}(u,1)}{f_d(u,1)}u\,,\ -\frac{f_{d-1}(u,1)}{f_d(u,1)}\right)$$

defines a birational function $\mathbb{A}^1_{\mathbb{C}} \dashrightarrow C$.

iii) Discuss point ii) (eventually through examples) in the case where f is reducible.

c) Let $C = \{(x,y) \in \mathbb{A}^2_{\mathbb{C}} \,|\, y^2 = x^2 + x^3\}$.

i) Determine the singular points of C and the corresponding multiplicities.

ii) Prove that C is rational and determine an explicit birational parametrization $\mathbb{A}^1_{\mathbb{C}} \dashrightarrow C$.

iii) Determine the automorphism group of C, $\mathrm{Aut}(C)$.

d) Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a plane projective algebraic curve and let $L \subset \mathbb{P}^2_{\mathbb{C}}$ be a line such that $L \notin C$.

i) Recall the definition of the intersection multiplicity between C and L at the point $P \in L$. Prove that the following equality holds:

$$\mathrm{mult}_P(C \cap L) = \mathrm{mult}_{T(P)}(T(C) \cap T(L))\,,$$

for any projective transformation $T\colon \mathbb{P}^2_{\mathbb{C}} \to \mathbb{P}^2_{\mathbb{C}}$.

ii) Prove the following equality:

$$\sum_{P \in L} \mathrm{mult}_P(C \cap L) = d\,,$$

where d is the degree of C.