Scuola Internazionale Superiore di Studi Avanzati, Trieste

Area of Mathematics

Selection for the PhD Course in Geometry and Mathematical Physics

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Each candidate is required to solve at least one of the following exercises. Every answer must be sufficiently motivated.

A. CLASSICAL MECHANICS

1. Consider a point particle of mass m and the position vector $\mathbf{x} \in \mathbb{R}^3$ (with Cartesian components $x_1,\ x_2,\ x_3$). A force $\mathbf{F}$ acts on the particle, with Cartesian components $F_i = -k_i x_i$, $1 \le i \le 3$. Here k_1, k_2, k_3 are real constants such that $k_i \neq k_j$ for $i \neq j$. This system is called anisotropic oscillator. Show that the motion of the point particle can be described by the Euler–Lagrange equations with the Lagrangian

$$L = \frac{1}{2} m \sum_{i=1}^{3} \dot{x}_i^2 - \frac{1}{2} \sum_{i=1}^{3} k_i x_i^2, \tag{1}$$

Write explicitly the solution to the Euler–Lagrange equations of motion.

2. Consider again the anisotropic oscillator described above, but now assume that the particle is *constrained* to move along the surface of the sphere $|\mathbf{x}|^2 \equiv \sum_{i=1}^{3} x_i^2 = r^2$, where $r > 0$ is a constant. Compute the constraint force (as a function of position and velocity) that keeps the particle on the sphere, assuming that the constraint force is normal to the surface.

3. Prove that the anisotropic oscillator constrained to the sphere is mathematically equivalent, for suitable r, to an axially symmetric top with anisotropic quadratic potential, defined below.

An axially symmetric top (Figure 1) is a rigid body rotating around a fixed point O with respect to a fixed reference frame with origin O. Let I_1, I_2, I_3 be the (positive) principal moments of inertia with respect to a principal axes system with origin O, rotating with the body. "Axially symmetric" means that $I_1 = I_2$. The top is described by the Lagrangian

$$\mathcal{L} := \frac{1}{2} I_1 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi} \cos \theta)^2 - V(\mathbf{x}),$$

$$\theta \in [0, \pi], \qquad \phi \in [0, 2\pi), \qquad \psi \in [0, 2\pi),$$

with anisotropic quadratic potential $V(\mathbf{x}) = \sum_{i=1}^{3} \alpha_i x_i^2$, where $\alpha_i \neq \alpha_j$ are constants and x_1, x_2, x_3 now denote the components of the position vector $\mathbf{x}$ of the center of mass G with respect to the fixed reference frame.

[*Hints:*

i) It may be convenient to use spherical coordinates to describe the anisotropic oscillator on the sphere as at the point **2.**

ii) The center of mass G lies on the x_3-axis of the principal axes system, represented in Figure 1 with letter $\mathbf{k}$. *Its position* $\mathbf{x}$ *is a function of the Euler angles θ, ϕ, ψ. Moreover, $|\overrightarrow{OG}|$ is constant.*]

4. Consider the following Hamiltonian

$$H = \frac{1}{2m|\mathbf{x}|^2}|\mathbf{L}|^2 + \frac{1}{2}\sum_{i=1}^{3} k_i x_i^2, \tag{2}$$

where the constants m, k_1, k_2, k_3 are as above. Here $\mathbf{x}$ is a position vector, $|\mathbf{L}|^2$ is the Euclidean norm of the cross product $\mathbf{L} = \mathbf{x} \times \mathbf{p}$, where $\mathbf{p} \in \mathbb{R}^3$ is the momentum canonically conjugated to $\mathbf{x}$.

i) Prove that the Hamiltonian flow $\dot{x}_i = \partial H/\partial p_i$, $\dot{p}_i = -\partial H/\partial x_i$, $1 \leq i \leq 3$, with given initial conditions $\mathbf{x} = \mathbf{x}_0$, is constrained to the surface $|\mathbf{x}|^2 = r^2$, where $r^2 := |\mathbf{x}_0|^2$.

ii) For the above initial conditions, rewrite

$$H = \frac{1}{2mr^2}|\mathbf{L}|^2 + \frac{1}{2}\sum_{i=1}^{3} k_i x_i^2\Big|_{|\mathbf{x}|^2 = r^2} . \tag{3}$$

Show that the Hamiltonian of the anisotropic oscillator with Lagrangian (1) and constraint $|\mathbf{x}|^2 = r^2$ is (3). [*Hint: It may be convenient to use spherical coordinates.*]

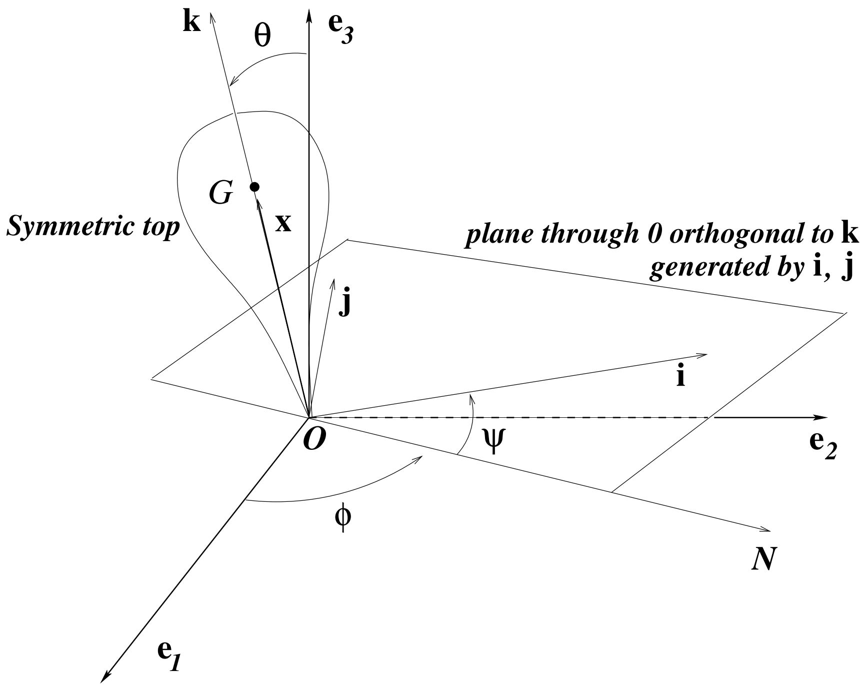

Figure 1: *Symmetric top. The orthonormal frame* O, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ *is the fixed reference frame. The orthonormal frame* O, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *is the frame of principal axes, rotating with the rigid body.* θ, ϕ *and* ψ *denote the* Euler angles *defined as follows. Let* N *be the oriented line of intersection between the horizontal plane through the origin* O *generated by* $\mathbf{e}_1$, $\mathbf{e}_2$, *and the plane through the origin normal to* $\mathbf{k}$. θ *is the oriented angle between* $\mathbf{e}_3$ *and* $\mathbf{k}$; ϕ *is the oriented angle between* $\mathbf{e}_1$ *and* N; ψ *is the oriented angle between* N *and* $\mathbf{i}$. *The position of the center of mass* G *is* $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$.

B. QUANTUM MECHANICS

Consider the following Schrödinger operator (the Hamiltonian for the Helium atom in normalised units):

$$H^{\mathrm{He}} = -\Delta_{\mathbf{x}_1} - \Delta_{\mathbf{x}_2} - \frac{2}{|\mathbf{x}_1|} - \frac{2}{|\mathbf{x}_2|} + \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|}$$

acting on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^3 \times \mathbb{R}^3, \mathrm{d}\mathbf{x}_1 \mathrm{d}\mathbf{x}_2)$. Note that H^{He} describes two *spinless* electrons moving around a nucleus with charge $Z = 2$ and, as a further simplification, H^{He} is *not* restricted to fermionic functions ("fermionic" = antisymmetric under exchange $\mathbf{x}_1 \leftrightarrow \mathbf{x}_2$; "bosonic" = symmetric under $\mathbf{x}_1 \leftrightarrow \mathbf{x}_2$). In this problem you shall discuss the approximate computation of the so-called "ground state energy" of H^{He}, namely the quantity

$$E_{\mathrm{GS}} := \inf_{\substack{\Psi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \\ \|\Psi\|_2 = 1}} \langle \Psi, H^{\mathrm{He}} \Psi \rangle .$$

($C_0^\infty \equiv$ functions that are infinitely differentiable and have compact support. $\|\Psi\|_2$ is the L^2-norm of Ψ.) The experimental value of this energy is $E_{\mathrm{GS}} = -78.8$ eV in *physical units*, which corresponds to $E_{\mathrm{GS}} = -1.45$ in the *normalised units* of this problem.

Recall that the Hamiltonian $\mathfrak{h} = -\Delta_{\mathbf{x}} - \frac{Z}{|\mathbf{x}|}$ of an Hydrogenoid atom with charge $-Z$, $Z > 0$, has ground state energy and ground state normalised eigenfunction respectively

$$E^{(Z)} = -\frac{Z^2}{4}, \qquad \varphi_Z(\mathbf{x}) = \frac{Z^{3/2}}{\sqrt{8\pi}} e^{-Z|\mathbf{x}|/2}, \qquad \mathbf{x} \in \mathbb{R}^3 .$$

(i) Assume first that the electron-electron interaction in H^{He} is absent, that is, consider

$$H_0^{\mathrm{He}} := -\Delta_{\mathbf{x}_1} - \Delta_{\mathbf{x}_2} - \frac{2}{|\mathbf{x}_1|} - \frac{2}{|\mathbf{x}_2|} .$$

Compute the quantity

$$E_0 := \inf_{\substack{\Psi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \\ \|\Psi\|_2 = 1}} \langle \Psi, H_0^{\mathrm{He}} \Psi \rangle .$$

(the "ground state energy" of H_0^{He}) and compare it with the experimental value of E_{GS}.

(ii) Compute the *upper bound* E_+ to E_{GS} that one obtains by using the bosonic trial function $\varphi_Z(\mathbf{x})\varphi_Z(\mathbf{y})$, where $Z > 0$ is a generic charge that you have to optimise to get the lowest upper bound possible. Compare E_+ with the experimental value of E_{GS}.

(iii) The optimal value $Z = Z_{\text{eff}}$ found in (ii) is actually smaller than 2, meaning that (within this variational approximation) each electron of the Helium atom is effectively subject to a nuclear charge $Z_{\text{eff}} < 2$. Give a concise physical interpretation of this phenomenon.

Does this phenomenon have anything to do with the stability of ordinary matter, and if so, in which sense? (Only a concise answer is requested.)

For the computation of an integral in part (ii) the following identity may be useful:

$$\frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|} = \begin{cases} \dfrac{1}{|\mathbf{x}_2|} \displaystyle\sum_{n=0}^{\infty} \left(\frac{|\mathbf{x}_1|}{|\mathbf{x}_2|}\right)^n P_n(\cos\theta) & \text{if } |\mathbf{x}_1| < |\mathbf{x}_2| \\ \dfrac{1}{|\mathbf{x}_1|} \displaystyle\sum_{n=0}^{\infty} \left(\frac{|\mathbf{x}_2|}{|\mathbf{x}_1|}\right)^n P_n(\cos\theta) & \text{if } |\mathbf{x}_2| < |\mathbf{x}_1| \end{cases}$$

where θ is the angle between $\mathbf{x}_1$ and $\mathbf{x}_2$, and

$$P_n(t) := \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left[(t^2-1)^n\right], \qquad t \in [-1,1], \quad n = 0,1,2,3,\dots,$$

is the n-th Legendre polynomial. Recall that P_n has the parity of n, that $P_0(t) = 1$, and that $\int_{-1}^{1} P_n(t) P_m(t)\, \mathrm{d}t = \frac{2}{2n+1}\delta_{nm}$.

C. ALGEBRAIC GEOMETRY

Let $\mathbb{P}^n$ be the n-dimensional projective space over a given field K. By definition $\mathbb{P}^n$ is the set of 1-dimensional vector subspaces of K^{n+1}. For any $(x_0, \ldots, x_n) \in K^{n+1} \setminus \{0\}$, we will denote by $x = (x_0 : \ldots : x_n) \in \mathbb{P}^n$ the corresponding point in $\mathbb{P}^n$. $x_0, \ldots, x_n$ are the homogeneous coordinates of $x \in \mathbb{P}^n$.

Any vector subspace $V \subset K^{n+1}$, with $V \neq \{0\}$, determines a subspace S of $\mathbb{P}^n$ in the following way:

$$S = \{x = (x_0 : \ldots : x_n) \in \mathbb{P}^n \mid (x_0, \ldots, x_n) \in V\}.$$

Such an S is called a *linear subspace* of $\mathbb{P}^n$. The dimension of S is defined to be $\dim_K V - 1$. A *line* in $\mathbb{P}^n$ is a linear subspace of dimension 1, while an *hyperplane* in $\mathbb{P}^n$ is a linear subspace of dimension $n-1$.

1) Consider the map

$$\begin{array}{rcl} \sigma_{n,m} \colon \mathbb{P}^n \times \mathbb{P}^m & \longrightarrow & \mathbb{P}^{nm+n+m} \\ (x_0 : \ldots : x_n; y_0 : \ldots : y_m) & \longmapsto & (x_0y_0 : \ldots : x_0y_m : x_1y_0 : \ldots : x_1y_m : \ldots : x_ny_m), \end{array}$$

where we use homogeneous coordinates z_{ij}, $i = 0, \ldots, n$, $j = 0, \ldots, m$, in $\mathbb{P}^{nm+n+m}$, hence $z_{ij}(\sigma_{n,m}(x; y)) = x_i y_j$. Prove that $\sigma_{n,m}$ is well defined, its image coincides with the set

$$\Sigma_{n,m} = \{z \in \mathbb{P}^{nm+n+m} \mid z_{ij}z_{k\ell} - z_{i\ell}z_{kj} = 0\,,\; i, k = 0, \ldots, n\,,\; j, \ell = 0, \ldots, m\}$$

and that it gives a bijection between $\mathbb{P}^n \times \mathbb{P}^m$ and $\Sigma_{n,m}$. The maps $\sigma_{n,m}$ are called *Segre maps.*

2) Consider $\Sigma_{n,m} \subset \mathbb{P}^{nm+n+m}$ with the structure of projective algebraic variety defined by the equations $z_{ij}z_{k\ell} - z_{i\ell}z_{kj} = 0$, $i, k = 0, \ldots, n$, $j, \ell = 0, \ldots, m$. Give $\mathbb{P}^n \times \mathbb{P}^m$ the structure of projective algebraic variety induced by $\Sigma_{n,m}$ via $\sigma_{n,m}$, in such a way that $\sigma_{n,m}$ is an isomorphism.

   Prove that the projections $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$, $(x; y) \mapsto x$, and $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$, $(x; y) \mapsto y$, are morphisms of algebraic varieties.

3) Show that $\Sigma_{n,m}$ is not contained in any proper linear subspace $S \subset \mathbb{P}^{nm+n+m}$.

4) Prove that, for any $y \in \mathbb{P}^m$, the image $\sigma_{n,m}(\mathbb{P}^n; y) \subset \mathbb{P}^{nm+n+m}$ is a linear subspace of dimension n. Similarly, for any $x \in \mathbb{P}^n$, $\sigma_{n,m}(x; \mathbb{P}^m) \subset \mathbb{P}^{nm+n+m}$ is a linear subspace of dimension m.

5) Consider now $\Sigma_{2,1} \subset \mathbb{P}^5$. Show that any line S in $\mathbb{P}^5$ which is contained in $\Sigma_{2,1}$, is contained in $\sigma_{2,1}(x; \mathbb{P}^1)$ for some $x \in \mathbb{P}^2$, or in $\sigma_{2,1}(\mathbb{P}^2; y)$ for some $y \in \mathbb{P}^1$.
   (Hint: show that, if $S \subset \Sigma_{2,1}$ is a line in $\mathbb{P}^5$ through $\sigma_{2,1}(x, y)$ and $\sigma_{2,1}(x', y')$, then either $x = x'$ or $y = y'$.)

D. DIFFERENTIAL GEOMETRY

For a given positive number R denote $S^n \subset \mathbb{R}^{n+1}$ the sphere of radius R in the $(n+1)$-dimensional Euclidean space,

$$S^n = \{\mathbf{x} = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_1^2 + \cdots + x_{n+1}^2 = R^2\}.$$

1) Introduce on S^n a structure of smooth n-dimensional manifold such that the natural map $S^n \hookrightarrow \mathbb{R}^{n+1}$ is an embedding of smooth manifolds.

2) Define a structure of smooth manifold on the Cartesian product of two spheres $S^m \times S^n$.

3) Calculate the cohomology ring $H^*(S^m \times S^n; \mathbb{R})$.

4) For any pair of positive integers m, n construct an embedding of the manifold $S^m \times S^n$ into Euclidean space $\mathbb{R}^{m+n+1}$.

E. COMPLEX ANALYSIS

Consider the following function of a complex variable $z \in \mathbb{C}$

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

1. Determine the domain of analyticity of $f(z)$.
2. Find the image of the locus $|z| = R$ discussing the cases $R > 1$, $R < 1$ and $R = 1$.
3. Find the image of the finite region of the complex plane bounded by the curves $z_1 = \rho$, $z_2 = \rho e^{i\frac{\pi}{4}}$, with $0 \leq \rho \leq 1$, and $|z_3| = 1$.
4. Determine the points for which the map $f(z)$ is not conformal; compute the factor by which angles between the tangents to two lines outgoing from such points get multiplied under the map.
5. For a generic analytic function $g(z)$, find the condition under which the angles between the tangents to two lines outgoing from a point z get multiplied by a factor $m \in \mathbb{N}$ under the map.