SISSA – Mathematics Area

Entrance examination for the course in Mathematical Analysis, Modelling,and Applications

September 10, 2019

Solve FIVE of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

Mathematical Analysis

1. Let $(a_n)_{n\in\mathbb{N}}$ be defined by
$$\begin{cases} a_{n+1} = a_n - a_n^2 \\ a_0 = 1/2 \end{cases}$$
   (a) Prove that $0 < a_{n+1} < a_n$ and that $a_n \to 0$.
   (b) Prove that $\lim_{n\to\infty} na_n = 1$.

2. Let $f(t,x)$, with $t, x \in \mathbb{R}$, be a smooth real function with the following properties:
$$f(t+1,x) = f(t,x), \quad \frac{\partial f}{\partial x}(t,x) > 0,$$
for all $t, x \in \mathbb{R}$. Show that the differential equation $\dot{x} = f(t,x)$ has at most one periodic solution.

3. For $h \in \mathbb{R}$ let $\tau_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the multiplication operator
$$\tau_h[f](x) = \cos(hx)f(x).$$
   (a) Prove that for all f in $L^2(\mathbb{R})$
$$\lim_{h\to 0} \|\tau_h f - f\|_{L^2(\mathbb{R})} = 0.$$
   (b) Prove that
$$\liminf_{h\to 0} \sup_{\|f\|_{L^2(\mathbb{R})}=1} \|\tau_h f - f\|_{L^2(\mathbb{R})} > 0.$$
   (c) Prove that for each function $\omega : [0,+\infty) \to (0,1]$ with $\lim_{t\to 0}\omega(t) = 0$ there exists $f \in L^2(\mathbb{R})$ such that
$$\liminf_{h\to 0} \frac{\|\tau_h f - f\|_{L^2(\mathbb{R})}}{\omega(|h|)} = +\infty.$$

4. Let $f(x)$, with $x \in \mathbb{R}$, be a continuous real function such that $f(f(x)) = x$, for all $x \in \mathbb{R}$. Show that f has a fixed point: there exists $x_0 \in \mathbb{R}$ such that $f(x_0) = x_0$.

5. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that
$$f(x) = x + o(x),$$
where
$$\lim_{|x|\to 0} \frac{o(x)}{|x|} = 0.$$
Prove or disprove the following statements
   (a) There exist U and V open neighborhoods of 0 such that $f : U \to V$ is injective.
   (b) There exist U and V open neighborhoods of 0 such that $f : U \to V$ is surjective.

6. Let $A_i \subset [0,1]$, $i = 1,\dots,n$ be Lebesgue measurable sets. Assume $\sum_{i=1}^n |A_i| > n-1$. Conclude that $|\bigcap_{i=1}^n A_i| > 0$.

7. Let $A, B \subset \mathbb{R}$ be measurable sets of positive measure. Show that $A - B$ contains an interval. [Hint: Consider the convolution between characteristic functions of A and B.]

8. Let $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ be the ball of radius r centered at the origin. Assume that $u \in C^2(B_3)$ is harmonic,
$$\Delta u = 0.$$
   (a) Let $\varphi : [0,1] \to \mathbb{R}$ be a continuous function; show that for all $x \in B_1$
$$u(x) \int_{B_1} \varphi(|y|)dy = \int_{|y-x|\le 1} \varphi(|x-y|)u(y)dy.$$
   (b) Use the previous result to show that $u \in C^\infty(B_1)$ and that for every $k \in \mathbb{N}$ there exists $C = C(k)$ such that for all $x \in B_1$
$$|\nabla^k u(x)| \le C \int_{B_2} |u|,$$
   where $\nabla^k u$ is the collection of all k-th derivatives of u.

9. Let $\Omega \in \mathbb{R}^n$ be an open set and let $u(x,t)$ be a smooth solution of the following initial boundary value problem:
$$\begin{cases} u_{tt} - \Delta u + u^3 = 0, & \text{for } (x,t) \in \Omega \times [0,T], \\ u(x,t) = 0, & \text{for } (x,t) \in \partial\Omega \times [0,T]. \end{cases}$$
   Show that if $u|_{t=0} = u_t|_{t=0} = 0$ for $x \in \Omega$, then $u \equiv 0$.
   [Hint: Derive an energy equality for u.]

10. The operator $K : L^2[-\pi,\pi] \to L^2[-\pi,\pi]$ is given by
$$(Kf)(t) = \int_{-\pi}^{\pi} |t-s| f(s)\, ds.$$
   (a) Prove that K is a compact and self-adjoint operator.
   (b) Find the eigenvalues of K.

Numerical Analysis

11. Let $A \in \mathbb{R}^{2\times 2}$, defined as
$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$
   be a symmetric and positive definite (SPD) matrix.
   (a) Provide a definition of an SPD matrix. Determine conditions, if any, on a, b and c such that A is SPD.
   (b) Define the Cholesky factorization of an SPD matrix. Derive analytically the expression of the Cholesky factor L of the matrix A.

   Now consider the following algorithm: given an SPD matrix $B \in \mathbb{R}^{n\times n}$

Algorithm 1 with input B

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Let B_0 := B
for k = 0, 1, 2, ... do
    Compute the Cholesky factorization L_k of B_k, i.e. B_k = L_k L_k^T.
    Define B_{k+1} := L_k^T L_k.
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   (c) Show that the Algorithm 1 is well defined, i.e. B_k is SPD for any $k \in \mathbb{N}$.

(d) Show that B_k is similar to B, i.e. that there exists an invertible matrix M_k such that $B_k = M_k^{-1} B M_k$.

(e) Apply one iteration of Algorithm 1 to the matrix A defined above.

12. Let $a \in \mathbb{R}^+$. Consider the following iterative method

$$x^{(k+1)} = \frac{1}{2}\left(x^{(k)} + \frac{a}{x^{(k)}}\right). \tag{1}$$

(a) Show that, for fixed a, there exists $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that (1) is the Newton method applied to f, by explicitely determining the expression of f.

(b) Show that (1) is globally convergent to a (unique) root α. Explicitely determine the expression of α (as a function of a).

(c) Describe a different method (other than the Newton method) to find the root α of f. Compare your method to the one in (1) e.g. in terms of convergence rate.

13. Let $\Omega = (0, 3)$, $\gamma : \Omega \to \mathbb{R}$. Consider the following diffusion equation: find $\phi : \Omega \to \mathbb{R}$, $\phi = \phi(x)$ such that

$$\begin{cases} -(\gamma\phi')' = 0 & \text{in } \Omega, \\ \phi(0) = 10, \\ \gamma(3)\phi'(3) = -2, \end{cases} \tag{2}$$

where $\phi' = \frac{\partial \phi}{\partial x}$.

(a) Choose one discretization technique (e.g. Finite Difference, Finite Element, or Finite Volume method). Write all the steps necessary to derive the discrete system of equations (as well as any assumption on problem data you might need in order to do that). Explicitely write the resulting system of equations when Ω is discretized into $n = 3$ non-overlapping subintervals.

(b) Describe which methods you would use to solve the system of equations resulting from the discretization in (a), differentiating in particular the limit cases of small n (e.g. $n = 3$) and very large values of n (i.e., $n \to \infty$).

Consider now the following advection-diffusion problem: find $\varphi : \Omega \to \mathbb{R}$, $\varphi = \varphi(x)$ such that

$$\begin{cases} -(\mu\varphi')' + a\varphi' = 0 & \text{in } \Omega, \\ \varphi(0) = 10, \\ \varphi(3) = 0, \end{cases} \tag{3}$$

where $\mu, a \in \mathbb{R}^+$.

(c) Repeat step (a) for the advection-diffusion problem (3), highlighting the most relevant differences with respect to the answer you provided in the diffusion case (2).

(d) Discuss under which conditions the solution of the discretized problem obtained in (c) might incur in stability issues, and propose possible numerical remedies.

14. Let $\Omega = (0, L)$, and consider the following linear advection problem in Ω: find $u : \Omega \to \mathbb{R}$, $u = u(x)$ such that

$$\partial_t u + \partial_x F(u) = 0 \quad \text{in } \Omega, \tag{4}$$

where $F : \mathbb{R} \to \mathbb{R}$,

$$F(u) = au, \tag{5}$$

and $a \in \mathbb{R}$.

(a) Impose suitable boundary and initial conditions.

(b) Partition Ω into n subintervals of equal length. Propose a discretization of the problem (based e.g. on Discontinuous Galerkin, Finite Difference, Finite Element, or Finite Volume schemes). Comment on the properties of the resulting scheme. Write all the steps necessary to derive the discrete system of equations.

(c) Suppose that the discrete system can be written in the following matrix form: $M\frac{d}{dt}\mathbf{u} = K\mathbf{u}$, where M and K are suitable matrices (which you are not required to derive from (b)) and $\mathbf{u}$ is the vector of unknowns. Perform a time discretization by means of the explicit Euler method.

(d) Describe the stability properties of the explicit Euler method for a generic system of ordinary differential equations.

15. Consider the steady Stokes problem: given $\Omega \subset \mathbb{R}^2$, find $(\mathbf{u}, p)$, $\mathbf{u} : \Omega \to \mathbb{R}^2$, $p : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}_{\text{in}} & \text{on } \Gamma_{\text{in}}, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_{\text{wall}}, \\ \mathbf{u} = \mathbf{u}_{\text{out}} & \text{on } \Gamma_{\text{out}}, \end{cases} \tag{6}$$

where Γ_{in}, Γ_{wall} and Γ_{out} form a partition of $\partial\Omega$.

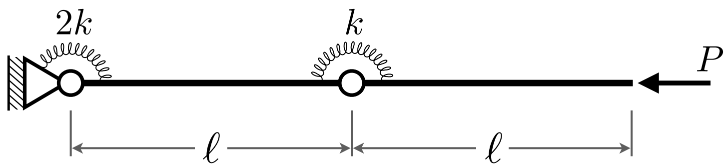
(a) Write the weak formulation of the problem (6), specifying function spaces for unknown velocity and pressure, as well as for the test functions. Study existence and uniqueness of the solution, introducing any required hypothesis.

(b) Let $\mathbf{u}_{\text{in}} : \Gamma_{\text{in}} \to \mathbb{R}^2$, $\mathbf{u}_{\text{out}} : \Gamma_{\text{out}} \to \mathbb{R}^2$ be given boundary data. Denoting by $\mathbf{n}$ the outward unit normal vector to $\partial\Omega$, prove that $\int_{\Gamma_{\text{in}}} \mathbf{u}_{\text{in}} \cdot \mathbf{n} = -\int_{\Gamma_{\text{out}}} \mathbf{u}_{\text{out}} \cdot \mathbf{n}$ is a necessary condition for the existence of a weak solution. Provide a physical interpretation of the aforementioned condition.

(c) Introduce a discretization scheme based on finite element method, noting any required hypotheses.

Continuum Mechanics

16. Compute the critical load P_c of the elastic, discrete system shown in the figure. Notice that the two torsional springs are of different stiffness, *i.e.* $2k$ and k.

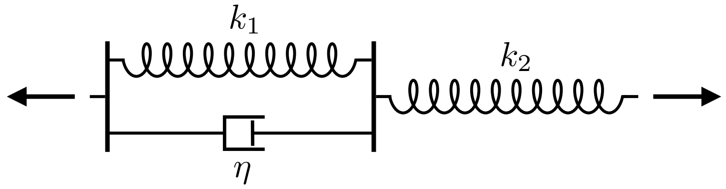


17. Consider the homogeneous deformation $\mathbf{y} : \mathcal{B} \to \mathcal{E}$ such that (in cartesian components)

$$\begin{aligned} y_1 &= x_1 + \gamma\, x_2\,, \\ y_2 &= \beta\, x_2\,, \\ y_3 &= x_3\,, \end{aligned}$$

where γ and β are positive scalars, and x_i, $i = \{1, 2, 3\}$, are the coordinates of a material point $\mathbf{x} \in \mathcal{B}$. Compute the principal stretches.

18. Compute the constitutive response (relation between applied force, elongation, and their time derivatives) of the one-dimensional rheological model shown in the figure. Notice that the system comprises two linear elastic springs and a linear dashpot.



19. A cylindrical body of height h and circular bases of radius r is in equilibrium under prescribed uniform tractions $\pm\sigma\boldsymbol{e}_z$ applied at its extremities ($\boldsymbol{e}_z$ denotes the cylinder axis) and constrained such that radial displacements are prohibited on its lateral surface. Compute the elongation of the cylinder and the magnitude of the lateral contact forces by exploiting linear elasticity theory and assuming isotropic material response.

20. Let $\mathbf{y}:\mathcal{B}\times\mathbb{R}\to\mathcal{E}$ be a plane motion represented in cartesian components by

$$
\begin{aligned}
y_1 &= x_1 \exp(t)\,,\\
y_2 &= x_2 + t\,,\\
y_3 &= x_3\,,
\end{aligned}
$$

where x_i, $i=\{1,2,3\}$, are the coordinates of a material point $\mathbf{x}\in\mathcal{B}$ and $t\geq 0$ is time. Compute the spatial description of the velocity field and determine the streamlines.