SISSA – Mathematics Area

Entrance examination for the course in Mathematical Analysis, Modelling, and Applications

September 2, 2013

Solve FIVE of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

Differential equations and measure theory

1. Find a solution u of the problem

$$\begin{cases} u_{tt}(t,x) = u_{xx}(t,x), & t \in \mathbb{R},\ x \in [0,\pi], \\ u(0,x) = 0, & x \in [0,\pi], \\ u_t(0,x) = \sin(x), & x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, & t \in \mathbb{R}. \end{cases}$$

2. Let $-\pi/2 < \lambda < \pi/2$. Consider the Cauchy problem

$$\begin{cases} y'(x) = \cos(y(x)), & x \in \mathbb{R}, \\ y(0) = \lambda. \end{cases}$$

(a) Prove that there is one and only one solution y defined on $\mathbb{R}$.

(b) Prove that

$$\lim_{x \to \pm\infty} y(x) = \pm\frac{\pi}{2}.$$

(c) Prove that there exists a unique $x_0 \in \mathbb{R}$ such that $y(x_0) = 0$.

(d) Prove that $y(x_0 - x) = -y(x_0 + x)$ for every $x \in \mathbb{R}$.

3. Let A be a measurable subset of $\mathbb{R}$. For every $f \in L^1(\mathbb{R})$ and for every $y \in \mathbb{R}$ let

$$T(f,y) := \int_A f(x-y)\,dx.$$

(a) Prove that for every $y \in \mathbb{R}$ the function $f \mapsto T(f,y)$ is continuous on $L^1(\mathbb{R})$.

(b) Prove that for every $f \in L^1(\mathbb{R})$ the function $y \mapsto T(f,y)$ is continuous on $\mathbb{R}$.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let u_n, f_n, v_n, u, f, v be measurable real functions on X, with $u_n \to u$, $f_n \to f$, and $v_n \to v$ a.e. in X. Suppose that for every n we have $u_n \leq f_n \leq v_n$ a.e. on X, that u_n, u, v_n, v are in $L^1(\mu)$, and that

$$\lim_{n\to\infty} \int_X u_n d\mu = \int_X u d\mu \qquad \text{e} \qquad \lim_{n\to\infty} \int_X v_n d\mu = \int_X v d\mu.$$

(a) Prove that $f_n, f \in L^1(\mu)$ and that

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

(b) Deduce from (a) that, if $f_n, f \in L^1(\mu)$, $f_n \to f$ a.e. in X, and $\|f_n\|_{L^1} \to \|f\|_{L^1}$, then we have also

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

Basic analysis and functional analysis

5. We say that a sequence x_n, $n = 1, 2, \ldots$, Cesaro converges to a, and we write $x_n \overset{c}{\to} a$, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = a.$$

A function f is Cesaro continuous at a if $x_n \overset{c}{\to} a$ implies $f(x_n) \overset{c}{\to} f(a)$. Prove that, if $f\colon \mathbb{R} \to \mathbb{R}$ is Cesaro continuous at 0 and $f(0) = 0$, then f is linear.
(Hint: every sequence of the form $x, y, z, x, y, z, \ldots$ is Cesaro convergent)

6. Prove that for every nonnegative integer n we have

$$\sum_{k=1}^{\infty} \left[\frac{n}{2^k} + \frac{1}{2} \right] = n\,,$$

where $[x]$ denotes the integer part of x.
(Hint: use the binary representation of n)

7. Let $T\colon C([0,1]) \to C([0,1])$ be the linear operator defined by $T(f) := y$, where y is the solution of the Cauchy problem

$$\begin{cases} y'(x) = y(x) + f(x), & x \in [0,1], \\ y(0) = 0. \end{cases}$$

(a) Prove that T is compact.

(b) Prove that T has no eigenvalues.

8. Consider the operator $T\colon C([0,1]) \to C([0,1])$ defined by

$$(Tf)(t) := \int_0^1 \kappa(t,s) f(s) ds,$$

where $\kappa : [0,1]^2 \to \mathbb{R}$ satisfies the following properties:

- for all $t \in [0,1]$, the function $\kappa_t(s) := \kappa(t,s)$ is integrable in s:
$$\int_0^1 \big|\kappa_t(s)\big| ds < \infty,$$
- the function $[0,1] \ni t \mapsto \kappa_t \in L^1([0,1])$ is continuous.

Show that T is compact.

9. Consider the subspace M of

$$\ell^\infty = \Big\{ x : \mathbb{N} \to \mathbb{R} : \|x\|_{\ell^\infty} = \sup_n |x(n)| < \infty \Big\}$$

defined by

$$M = \Big\{ x \in \ell^\infty : x(n) \neq 0 \text{ only for a finite number of } n \in \mathbb{N} \Big\}.$$

(a) Verify if M is closed or explicitly write $\bar{M}$.

(b) Show that it is not possible to find a linear functional $f \in (\ell^\infty)^*$ such that

$$\text{Kernel} \, f = \bar{M}.$$

(c) Show that the set

$$K := \bar{M} \cap \big\{ \|x\|_{\ell^\infty} \leq 1 \big\}$$

does not have extremal points.

10. Let $J \subset [0,1]$ be closed and consider a closed subspace E of $C([0,1])$. Assume that for all $g \in C(J)$ there exists $f \in E$ such that

$$f(t) = g(t) \text{ for all } t \in J.$$

Show that one can find a constant c independent of g such that the following statement holds:

for all $g \in C(J)$ there exists $f \in E$ such that $f(t) = g(t)$ e $\|f\|_{C([0,1])} \leq c \|g\|_{C(J)}$.

Numerical analysis

11. Denote $V = H^1_0(0,1), a : V \times V \to \mathbb{R}$ a bilinear form and $F : V \to \mathbb{R}$ a linear form defined as follows:

$$a(u,v) \equiv \int_0^1 \frac{1}{1+x}\, u'(x)\, v'(x)\, dx, \qquad F(v) \equiv \int_0^1 \left(-e^x - \frac{1}{(1+x)^2}\right) v(x)\, dx.$$

Moreover, denote u the solution of the following problem, under weak form:

$$\text{find} \;\; u \in V \;\; \text{such that} \quad a(u,v) = F(v), \quad \forall v \in V.$$

1. Consider the Galerkin finite element approximation of ():

$$\text{find} \;\; u_h \in V_h : \quad a(u_h, v_h) = F(v_h) \qquad \forall v_h \in V_h,$$

   where V_h is a subspace of V with finite dimension, which depends on the positive real parameter h. Provide an a priori estimation of the error $\|u - u_h\|_V$ with respect to the data of the problem, by assuming that

$$\inf_{w_h \in V_h} \|u - w_h\|_V \leq Ch|u|_{H^1(0,1)},$$

   with a constant C independent of h and v.

12. Consider the following boundary value problem:

$$\begin{aligned} &-(\mu u')' + \beta u' + \gamma u = f \quad \text{in } \Omega = (a,b) \\ &u(a) = 0 \\ &u'(b) = g_b \end{aligned}$$

where we assume that $g_b \in \mathbb{R}$, $\mu, \beta, \beta', \gamma \in C^0([a,b])$, and $f \in L^2(a,b)$.

1. Write the problem under weak form by making explicit the definition of the space V, of the bilinear form $a(\cdot,\cdot)$, and of the linear form $F(\cdot)$.
2. Give some conditions for the bilinear form $a(\cdot,\cdot)$ and $F(\cdot)$ to be coercive.

13. Let us denote by $A \in \mathbb{R}^{n\times n}$ a symmetric and positive definite matrix and by $\mathbf{b} \in \mathbb{R}^n$ a given vector.

1. Give a definition of the *gradient method* (also called nonstationary Richardson method) to solve the linear system $A\mathbf{x} = \mathbf{b}$, by explaining the meaning of its name and motivating the choice of the descent directions.

2. Show that the acceleration parameter α_k is given by the solution of the following minimization problem:

$$\Phi(\mathbf{x}^{(k)} + \alpha_k \mathbf{r}^{(k)}) = \min_{\alpha \in \mathbb{R}} \Phi(\mathbf{x}^{(k)} + \alpha \mathbf{r}^{(k)}),$$

where $\mathbf{r}^{(k)}$ is the residual at the step k and $\Phi(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A \mathbf{x} - \mathbf{b}^T \mathbf{x}$ is the so-called energy functional associated to the linear system $A\mathbf{x} = \mathbf{b}$.

14. Let $A \in \mathbb{R}^{2\times 2}$ be the matrix whose elements are $a_{11} = a_{22} = 1$, $a_{12} = \gamma$, $a_{21} = 0$, and

$$K_p(A) = \|A\|_p \|A^{-1}\|_p$$

the condition number (with respect to the p- norm) of A. We remind that the p-norm for a matrix A is given by

$$\|A\|_p := \sup_{\|x\|_p = 1} \|Ax\|_p \,, \qquad \text{dove} \qquad \begin{cases} \|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}\,, & \text{se } 1 \le p < \infty, \\ \max\{|x_1|, |x_2|\}, & \text{se } p = \infty. \end{cases}$$

Verify that, for any $\gamma \ge 0$,

$$K_\infty(A) = K_1(A) = (1+\gamma)^2.$$

Let $\mathbf{b} \in \mathbb{R}^2$ and let $\mathbf{x}$ be the solution to the linear system $A\mathbf{x} = \mathbf{b}$. Prove that

$$\frac{\|\delta\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} \le (1+\gamma)^2 \frac{\|\delta\mathbf{b}\|_\infty}{\|\mathbf{b}\|_\infty},$$

where $(\mathbf{x} + \delta\mathbf{x})$ is the solution of the perturbed system $A(\mathbf{x} + \delta\mathbf{x}) = (\mathbf{b} + \delta\mathbf{b})$, where $\delta\mathbf{b}$ is a perturbation of the vector $\mathbf{b}$.

15. Let g be a continuous function defined over the interval $[-1, 1]$. Let us consider three interpolation points $t_0 = -1$, $t_1 = \alpha$ and $t_2 = 1$, being α a real number such that $|\alpha| < 1$. In order to approximate the integral $\int_{-1}^{1} g(t)dt$, let us consider the following quadrature formula:

$$I_2(g) = \sum_{j=0}^{2} \omega_j g(t_j) = \omega_0 g(-1) + \omega_1 g(\alpha) + \omega_2 g(1).$$

1. Find the weights $\omega_0, \omega_1, \omega_2$ as functions of α, such that the formula is exact of degree 2.
2. Moreover, find α such that $I_2(p) = \int_{-1}^{1} p(t)dt$ for any polynomial function of degree 3.