**Scuola Internazionale Superiore di Studi Avanzati, Trieste**

**Admission Exam for the Phd Course in Mathematical Analysis, Modelling and Applications**

**Written exam, 10 September 2015**

The candidate should solve FIVE among the following exercises and should clearly mark on the first page of the solutions which exercises have been solved and have to be corrected (no more than FIVE exercises, please).

**Ex. 1.** Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function.

(a) Prove that the limits $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} f(x)$ exist.

(b) Prove that if both the limits $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} f(x)$ are finite, then the function $f$ is constant.

**Ex. 2.** Consider a not identically zero function $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) := \sum_{i=1}^{n} a_i e^{xb_i},$$

where $a_i, b_i \in \mathbb{R}$, $i = 1, \dots, n$. Show that the equation $f(x) = 0$ has at most $n - 1$ solutions.
(hint: proceed by induction)

**Ex. 3.** Let $p \in (1, \infty)$ and $f_n, f_\infty \in L^p(0, 1)$, $n \in \mathbb{N}$.

(a) Prove that $(f_n)$ weakly converges to $f_\infty$ in $L^p(0, 1)$ if and only if $\sup_n \|f_n\|_{L^p} < \infty$ and for every $x \in (0, 1)$ we have

$$\lim_{n\to\infty} \int_0^x f_n(y)\,\mathrm{d}y = \int_0^x f_\infty(y)\,\mathrm{d}y.$$

(b) Does the equivalence above remain true if we drop the condition $\sup_n \|f_n\|_{L^p} < \infty$?

**Ex. 4.** Let $p_1, p_2, q_1, q_2 \in (1, \infty)$ be such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$ for $i = 1, 2$. Define the spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ as:

$$E := L^{p_1}(\mathbb{R}) \cap L^{p_2}(\mathbb{R}), \qquad \|f\|_E := \max\left\{\|f\|_{L^{p_1}(\mathbb{R})}, \|f\|_{L^{p_2}(\mathbb{R})}\right\}$$

and

$$F := \left\{g_1 + g_2 \ : \ g_1 \in L^{q_1}(\mathbb{R}),\ g_2 \in L^{q_2}(\mathbb{R})\right\}, \qquad \|g\|_F := \inf\left\{\|g_1\|_{L^{q_1}(\mathbb{R})} + \|g_2\|_{L^{q_2}(\mathbb{R})}\right\},$$

where the inf is taken among all couples $g_1, g_2$ with $g_1 \in L^{q_1}(\mathbb{R})$, $g_2 \in L^{q_2}(\mathbb{R})$ such that $g = g_1 + g_2$.

(a) Prove that $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are Banach spaces.

(b) Let $L_1 : F \to \mathbb{R}$ be linear and continuous. Prove that there exists a unique $f \in E$ such that $L_1(g) = \int_{\mathbb{R}} fg \, \mathrm{d}\mathcal{L}$ for every $g \in F$, $\mathcal{L}$ being the Lebesgue measure, and that the operator norm of $L_1$ is equal to $\|f\|_E$.

(c) Let $L_2 : E \to \mathbb{R}$ be linear and continuous. Prove that there exists a unique $g \in F$ such that $L_2(f) = \int_{\mathbb{R}} fg \, \mathrm{d}\mathcal{L}$ for every $f \in E$ and that the operator norm of $L_2$ is equal to $\|g\|_F$.

**Ex. 5.** Let $(X, d)$ be a complete metric space and $F_n, F_\infty : X \to X$, $n \in \mathbb{N}$ be contractions with the same constant $\alpha \in (0, 1)$, i.e. such that

$$d\big(F_i(x), F_i(y)\big) \le \alpha \, d(x, y), \qquad \forall x, y \in X, \ i \in \mathbb{N} \cup \{\infty\}.$$

Assume that $F_n \to F_\infty$ pointwise and let $x_n, x_\infty \in X$ be such that $x_n = F_n(x_n)$ and $x_\infty = F_\infty(x_\infty)$, $n \in \mathbb{N}$. Prove that $x_n \to x_\infty$.

**Ex. 6.** Let $f : [0, \alpha] \to \mathbb{R}$ be a solution of the Cauchy problem

$$\begin{cases} f'(t) = f(t)^2 + t, \\ f(0) = 0. \end{cases}$$

Prove that $\alpha < 3$.

**Ex. 7.** Let $U : \mathbb{R}^2 \to \mathbb{R}$ be defined as $U(x_1, x_2) := x_1^2 + x_2^2 + a x_1 x_2$ with $a \in \mathbb{R}$. For which values of $a \in \mathbb{R}$ all the solutions of the differential equation

$$\ddot{x} = -\nabla U(x)$$

with $\mathbb{R} \ni t \mapsto x(t) \in \mathbb{R}^2$ are periodic?

**Ex. 8.** Let $B$ be a Banach space and let $\gamma : [0, 1] \to B$ be a curve continuous with respect to the weak topology.

(a) Assume that $B$ is separable. Is $\gamma$ Borel with respect to the Borel structure induced by the strong topology?

(b) Does the answer to the above question change if we drop the separability assumption?

**Ex. 9.** Let $a : [0,1] \to [1,2]$ be a Borel function and for every $f \in L^2(0,1)$ consider the functional $E_f : L^2(0,1) \to [0,+\infty]$ defined as

$$E_f(g) := \begin{cases} \displaystyle\int_0^1 \left(a|g'(x)|^2 + |f(x) - g(x)|^2\right) \mathrm{d}x, & \text{if } g \in W^{1,2}(0,1), \\ +\infty, & \text{otherwise.} \end{cases}$$

(a) Prove that for every $f \in L^2(0,1)$ there exists a unique minimum $T(f) \in L^2(0,1)$ of the functional $E_f$.

(b) Prove that $T : L^2(0,1) \to L^2(0,1)$ is a linear and continuous operator.

(c) Prove that $T : L^2(0,1) \to L^2(0,1)$ is a compact operator.

**Ex. 10.** For $f, g \in C([0,1])$ define

$$d(f,g) := \int_0^1 \min\{1, |f(x) - g(x)|\} \, \mathrm{d}x$$

(a) Prove that $d$ is a distance on $C([0,1])$.

(b) Let $L : C([0,1]) \to \mathbb{R}$ be a linear operator, continuous with respect to the topology induced by $d$. Prove that $L = 0$.

**Ex. 11.** Consider the Cauchy problem

$$y' = f(t,y), \quad y(0) = y_0,$$

and the following explicit two-stage Runge-Kutta method for its approximation:

$$y_{n+1} = y_n + h[\alpha_1 f(t_n, y_n) + \alpha_2 f(t_n + \theta h, y_n + \beta h f(t_n, y_n))],$$

with $\alpha_1$, $\alpha_2$, $\theta$ and $\beta$ given real coefficients.
Recall that the quantity $\bar{\lambda} = h\lambda$ is said to belong to the region of absolute stability if the approximate solution $\{y_n\}_{n=0}^{\infty}$ obtained by applying the scheme to the model differential equation $y' = \lambda y$ is bounded.

(a) Let $\alpha_1 = 1, \alpha_2 = 0$ (Explicit Euler method). Determine the region of absolute stability for real $\bar{\lambda}$, and indicate the order of the local truncation error of the method;

(b) Find the relationships between the coefficients $\alpha_1$, $\alpha_2$, $\theta$ and $\beta$ that ensure that the resulting approximation scheme is second order accurate;

(c) Determine the region of absolute stability for real $\bar{\lambda}$ in the general case.

**Ex. 12.** Let $\Pi_k$ be the set of polynomials of degree less than or equal to $k$. Given $q \in \Pi_{n+1}$ such that

$$\int_a^b p(x)q(x)\,\mathrm{d}x = 0 \quad \text{for all} \ \ p \in \Pi_n,$$

(a) show that the roots of $q$ are all simple and belong to the open interval $(a, b)$;

(b) Let $a = -1, b = 1$. Construct $p_i \in \Pi_i$, $i = 0, 1, 2$, such that $p_i(1) = 1$ and $\int_{-1}^{1} p_i(x)p_j(x)\,\mathrm{d}x = 0, \ \ i \neq j$;

(c) Find the quadratic polynomial $q_2(x)$ such that $\int_{-1}^{1} |x^3 - q_2(x)|^2\,\mathrm{d}x$ is minimal.

**Ex. 13.** Consider the linear algebraic system $Au = f$ on a finite dimensional linear vector space $V$, where $A$ is symmetric positive definite with respect to some inner product $(\cdot, \cdot)$. One version of the conjugate gradient method can be written as follows:

Given $u_0$, let $r_0 = f - Au_0$, $p_0 = r_0$.
For $k = 1, 2, \ldots$
$\alpha_k = (r_{k-1}, r_{k-1})/(Ap_{k-1}, p_{k-1})$,
$u_k = u_{k-1} + \alpha_k p_{k-1}$,
$r_k = r_{k-1} - \alpha_k A p_{k-1}$,
$\beta_k = (r_k, r_k)/(r_{k-1}, r_{k-1})$,
$p_k = r_k + \beta_k p_{k-1}$.

This method is known to satisfy the convergence estimate

$$\|u - u_k\|_A \leq 2\left(\frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1}\right)^k \|u - u_0\|_A$$

where $\|v\|_A^2 := (Av, v)$, and the condition number $\kappa(A)$ is defined as $\kappa(A) := \lambda_{\max}(A)/\lambda_{\min}(A)$ with $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ the maximum and minimum eigenvalues of $A$ respectively.
Let $B : V \to V$ be another symmetric positive definite operator with respect to the inner product $(\cdot, \cdot)$.

(a) Prove that $BA$ is symmetric positive definite with respect to the new inner product $(u, v)_{B^{-1}} := (B^{-1}u, v)$, $u, v \in V$.

A preconditioned conjugate gradient method can be obtained by applying the conjugate gradient method to the preconditioned system $BAu = Bf$ with respect to the new inner product. Using the above given algorithm for the conjugate gradient method and its convergence estimate:

(b) Derive the corresponding algorithm for the preconditioned conjugate gradient method. Your algorithm should not require evaluation of $B^{-1}$.

(c) Derive the corresponding convergence estimate in terms of the condition number of $BA$.

**Ex. 14.** Let us consider the problem: find the root $\alpha \in I = [0, 2]$ such that

$$\alpha = 2 - \mathrm{e}^{-\alpha}. \tag{1}$$

(a) Verify if it is possible to apply the bisection method to solve problem (1). In the case there is this possibility, estimate the number of necessary iterations to approximate $\alpha$ with a tolerance smaller than $10^{-3}$.

(b) Write a fixpoint scheme to solve problem (1) and analyze local and global convergence.

(c) Prove the a priori estimate for the error:

$$|x_k - \alpha| \leq C^k |x_0 - \alpha|.$$

(d) Introduce the stopping criterion

$$|x_{k+1} - x_k| \leq \varepsilon.$$

and show how this criterion implies the following error estimate:

$$|x_k - \alpha| \leq \frac{\varepsilon}{1 - C}.$$

**Ex. 15.** Let us consider the steady Stokes problem: find $(\mathbf{u}, p)$ such that

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = 0 & \text{in } \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{g} & \text{on } \mathit{\Gamma}_D \equiv \partial\Omega. \end{cases} \tag{2}$$

(a) Write the weak formulation of the problem (2), specifying the functional spaces for velocity and pressure, as well as for the test functions.

(b) Introduce the necessary compatibility conditions for the data $\mathbf{g}$ with the proper functional spaces to guarantee the well-posedness of the problem in the variational formulation.

(c) Find a Poiseuille analytical solution for velocity and pressure in a domain $\Omega = (x, y), x = [0, L], y = [0, 1]$ by introducing as boundary condition $\mathbf{g} = (y(y-1), 0)$.

**Ex. 16.** Cauchy's stress tensor is one of the key constructs in the classical mechanics of continuous media. Under suitable hypotheses, it is a symmetric tensor.

(a) Discuss the physical meaning of this symmetry property.

(b) Provide one concrete example of a physical system in which this property is satisfied and one in which it is not satisfied.

(c) State hypotheses that guarantee that Cauchy's stress tensor is symmetric, and give a proof of this symmetry property.