Scuola Internazionale Superiore di Studi Avanzati, Trieste

Entrance examination for the grants for "Laurea Magistrale in Matematica"

Written exam: 4 settembre 2014

The candidate is required to solve five of the following exercises. He has to choose at least one exercise in group A (exercises 1-5) and at least one in group B (exercises 6-10). Denote in a clear way on the first page which are the chosen exercises which must be evaluated (in any case no more than five).

Group A

1. Let $(\, , \,)$ be the euclidean scalar product on $\mathbb{R}^n$ and let ω_n be the measure of the unit sphere.

a) Let $A = (a_{ij})_{i,j=1}^n$ a matrix with real entries and let $Tr(A)$ its trace. Prove that

$$\int_{|x|<1} (Ax, x)\, dx = \frac{\omega_n}{n(n+2)} Tr(A).$$

b) Prove that if $u \in C_0^2(\mathbb{R}^n)$ then

$$\int_{\mathbb{R}^n} (\Delta u)^2 dx = \sum_{i,j=1}^n \int_{\mathbb{R}^n} |D_{ij}u|^2 dx.$$

2. Let u be an harmonic function on $\mathbb{R}^n$. Prove that

a) for every $p \geq 1$ the function $v := |u|^p$ is subharmonic in the sense of the mean, i.e., for every $R > 0$ and any $x \in \mathbb{R}^n$, we have

$$|u(x)|^p \leq \frac{1}{|B_R(x)|} \int_{B_R(x)} |u(y)|^p \, dy.$$

b) $w = |\nabla u|^2$ is subharmonic in the classical sense, i.e.

$$\Delta w \geq 0 \quad \text{on} \quad \mathbb{R}^n.$$

3. Let $h : (0, +\infty) \to \mathbf{R}$ be a continuous function, bounded, strictly decreasing with $h(1) = 0$. Let $g : \mathbf{R} \to \mathbf{R}$ be a positive function of class C^1 with bounded derivative g'. Consider the solution of the Cauchy problem

$$\begin{cases} \dot{x} = h(t)g(x) \\ x(1) = 2. \end{cases}$$

Prove that:

a) the solution $x(t)$ is defined for any $t \in (0, +\infty)$,

b) both $\lim_{t\to 0^+} x(t)$ and $\lim_{t\to+\infty} x(t)$ exist,

c) $\lim_{t\to 0^+} x(t)$ is finite and $\lim_{t\to+\infty} x(t) = -\infty$.

4. Let $I = (-1, 1)$ and let $\theta : I \to [0, +\infty)$ be a function of class $C^\infty(I)$ with the following properties:

- $\theta(x) = 0$ if and only if $x = 0$,
- $\theta''(0) > 0$.

a) Show that θ can be written as $\theta = \delta^2$ in I, where $\delta \in C^1(I)$ has the following properties:

  - $\delta(0) = 0$, $\delta'(0) > 0$,
  - $\delta(x) < 0$ for any $x \in I$ with $x < 0$, and $\delta(x) > 0$ for any $x \in I$ with $x > 0$.

b) Find an explicit expression of δ in terms of θ.

5. Let $\Gamma \subset \mathbb{R}^2_{(x_1,x_2)} \times \mathbb{R}_z$ be the surface defined by

$$\Gamma = \left\{ (x_1, x_2, z) : x_2 = \frac{1}{2}\left(z^3 - 3x_1 z\right) \right\}.$$

a) Write in parametric form, in a neighbourhood of the origin, the set S of all points of Γ where the tangent plane to Γ contains the e_3 direction (i.e., the z direction). Is the set S a smooth one-dimensional manifold?

b) Describe and draw the orthogonal projection of the set S on the plane $\mathbb{R}^2_{(x_1,x_2)}$.

c) Describe and draw the orthogonal projection of the set S on the plane $\mathbb{R}^2_{(x_1,z)}$.

Group B

6. Consider the problem

$$\begin{cases} \ddot{x} + (1 + c^2)x - 2c^2 x^3 = 0 \\ x(0) = 0 \\ \dot{x}(0) = 1 \end{cases}$$

where $c \in [0, 1]$. Prove that for any $c \in [0, 1)$ the solution $x_c(t)$ is a periodic function.

7. Let p be a prime number and let $\bar{a} \in \mathbb{Z}_p$, $\bar{a} \neq \bar{0}$.

a) Prove that the map

$$\phi : \mathbb{Z}_p \to \mathbb{Z}_p, \quad \phi(\bar{x}) = \bar{a}\bar{x}$$

is injective.

b) Deduce that any element $\bar{a} \neq \bar{0}$ in $\mathbb{Z}_p$ has an inverse element with respect to the product.

c) Let n be a natural number. Which elements $\bar{a} \in \mathbb{Z}_n$ have an inverse with respect to the product? (justify the answer).

8. For any $a \in \mathbb{C}$, prove the existence of a solution $\bar{z}$ of the equation

$$a\, z^2 - z + 1 = 0$$

such that

$$|\bar{z} - 1| \leq 1.$$

9.

a) Let $f : V \to V$ be a linear map of a finite dimensional vector space V and let W be a subspace of V such that $f(W) \subset W$. If f is triangularisable, prove that also its restriction $f|W : W \to W$ is triangularisable.

b) Let $f, g : V \to V$ be unitary automorphisms of a finite dimensional unitary space V, which commute. Prove the existence of an orthonormal basis of V formed by eigenvectors both for f and g.

10. Prove that

a) a metric space with a countable dense subset has a countable basis,

b) a compact metric space X has a countable basis.