SISSA – Mathematics Area

Entrance examination for the course in Mathematical Analysis, Modelling, and Applications

September 12, 2016

Solve FIVE of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

Measure theory

1. Let f_k, $f \in L^1(0,1)$, with $f_k \geq 0$, $f \geq 0$ a.e. in $(0,1)$. Suppose that $f_k \to f$ pointwise a.e. in $(0,1)$ and that

$$\int_0^1 f_k \, dx \to \int_0^1 f dx \, .$$

For every a, $b \in \mathbb{R}$ we set $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$.

(a) Prove that $f_k \wedge f \to f$ in $L^1(0,1)$.

(b) Prove that $\int_0^1 (f_k \vee f) \, dx + \int_0^1 (f_k \wedge f) \, dx \to 2 \int_0^1 f dx$.

(c) Prove that $f_k \vee f \to f$ in $L^1(0,1)$.

(d) Prove that $f_k \to f$ in $L^1(0,1)$.

2. Let $(f_n)_n$ be a bounded sequence in $L^3(\mathbb{R})$, such that $f_n \to f$ in $L^{3/2}(\mathbb{R})$. Prove that $f_n \to f$ in $L^2(\mathbb{R})$.

3. Let $f \in L^1(\mathbb{R}^n)$, with $f \geq 0$ a.e. in $\mathbb{R}^n$ and let $g \colon \mathbb{R}^n \to [0, +\infty)$ be the function defined by

$$g(x) := \int_{B(x,|x|/2)} f(y) \, dy \, ,$$

where $|\cdot|$ denotes the la Euclidean norm and $B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$ for every $x \in \mathbb{R}^n$ and every $r \geq 0$.

(a) Prove that g is continuous.

(b) Prove that $g(x) \to 0$ as $|x| \to \infty$.

(c) Prove that g has a maximum point in $\mathbb{R}^n$.

Differential equations

4. Consider the Cauchy problem

$$\begin{cases} y'(t) = \dfrac{t^2}{2+\sin(t^2)}\cos(y(t)^2) \\ y(0) = 0 \end{cases}$$

(a) Prove that there exist one and only one solution defined for every $t \in \mathbb{R}$.

(b) Prove that $y(t) \to \sqrt{\pi/2}$ as $t \to +\infty$.

5. Let $B := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $\partial B := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Find a function $u \in C^2(B) \cap C^0(\overline{B})$ such that

$$\begin{cases} \Delta u(x, y) = 0 & \text{per } (x, y) \in B\,, \\ u(x, y) = x^2 & \text{per } (x, y) \in \partial B\,, \end{cases}$$

where $\Delta u(x, y) = \dfrac{\partial^2 u}{\partial x^2}(x, y) + \dfrac{\partial^2 u}{\partial y^2}(x, y)$.

6. Let $u \in C^2(\mathbb{R})$ be a solution to the differential equation

$$-u''(t) + u(t) = u(t)^3\,, \qquad t \in \mathbb{R}. \tag{1}$$

(a) Prove that there exists a constant $C \in \mathbb{R}$ such that

$$\frac{1}{2}|u'(t)|^2 - \frac{1}{2}|u(t)|^2 + \frac{1}{4}|u(t)|^4 = C \qquad \text{for every } t \in \mathbb{R}. \tag{2}$$

(b) Prove that $C = 0$, provided that in addition u satisfies

$$\lim_{|t|\to\infty} u(t) = 0. \tag{3}$$

(c) Prove that every even solution to (1) satisfying (3) also satisfies $|u(0)|^2 = 2$ or $u(0) = 0$.

(d) Prove that that there exist exactly 3 even solutions to (1) with boundary conditions (3).

We recall that a function u is said to be even if $u(t) = u(-t)$ for every $t \in \mathbb{R}$.

7. For every $\alpha \in \mathbb{R}$ consider the two point problem

$$\begin{cases} y''(t) = \sin(\alpha y(t)) + (\cos(\alpha) - 1)\sin(t)\,, \\ y(0) = 0, \quad y(1) = 0\,, \end{cases} \tag{4}$$

and for every $\alpha \in \mathbb{R}$ and every $\beta \in \mathbb{R}$ consider the Cauchy problem

$$\begin{cases} y''(t) = \sin(\alpha y(t)) + (\cos(\alpha) - 1)\sin(t)\,, \\ y(0) = 0, \quad y'(0) = \beta\,. \end{cases}$$

(a) Prove that for every $\alpha \in \mathbb{R}$ and every $\beta \in \mathbb{R}$ there exists a unique solution $y_{\alpha,\beta}$ of the Cauchy problem defined for every $t \in \mathbb{R}$.

(b) Compute $y_{\alpha,\beta}(t)$ for $\alpha = 0$.

(c) Knowing that $\alpha \mapsto y_{\alpha,\beta}(t)$ is continuous in $\mathbb{R}$ for every β, $t \in \mathbb{R}$, prove that for $|\alpha|$ small enough there exists at least a solution of problem (4).

Functional analysis

8. Let $T\colon L^2(0,1) \to L^2(0,1)$ be the linear operator defined by

$$(Tu)(x) = \int_0^x u(t)\,dt \quad \text{for every } x \in (0,1)\,.$$

(a) Prove that T is compact.

(b) Find the spectrum of T.

9. Let $\psi \in L^2(0,1)$ and let

$$K_\psi = \{u \in L^2(0,1) : u \geq \psi \ \text{ a.e. in } (0,1)\ \}\,.$$

(a) Prove the minimization problem

$$\min_{u \in K} \int_0^1 |u|^2 dx$$

has a unique solution $u_\psi \in K_\psi$

(b) Find u_ψ explicitly.

10. Let V be a Banach space, $X \subset V$ be a weakly closed subset, and $Y \subset V$ be a weakly compact subset. Assume that $X \cap Y = \emptyset$ and let $d(X,Y) = \inf\{\|x - y\|\,, x \in X, y \in Y\}$.

(a) Prove that $d(X,Y) > 0$.

(b) Is the conclusion of the first point still true if we only assume that Y is weakly closed? If yes, provide a proof; if not, exhibit a counterexample.

Numerical analysis

11. Let $\Omega \subset \mathbb{R}^2$ be a domain with a smooth boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, where Γ_D and Γ_N are unions of a finite number of arcs and $\Gamma_D \cap \Gamma_N = \emptyset$. By introducing appropriate functional spaces, write the weak formulation of the following linear elasticity problem

$$\begin{cases} -\sum_{j=1}^{2} \frac{\partial}{\partial x_j}\sigma_{ij}(\mathbf{u}) = f_i & \text{in } \Omega, i = 1,2, \\ u_i = 0 & \text{on } \Gamma_D, i = 1,2, \\ \sum_{j=1}^{2} \sigma_{ij}(\mathbf{u})n_j = g_i & \text{on } \Gamma_N, i = 1,2, \end{cases} \tag{5}$$

having denoted $\mathbf{n} = (n_1, n_2)^T$ the outward unit normal vector to $\partial\Omega$, by $\mathbf{u} = (u_1, u_2)^T$ the unknown vector, and by $\mathbf{f} = (f_1, f_2)^T$ and $\mathbf{g} = (g_1, g_2)^T$, two assigned vector functions. Moreover, for $i, j = 1, 2$, let

$$\sigma_{ij}(\mathbf{u}) = \lambda \operatorname{div}(\mathbf{u})\delta_{ij} + 2\mu\epsilon_{ij}(\mathbf{u}), \epsilon_{ij}(\mathbf{u}) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big),$$

where λ and μ are positive constants and δ_{ij} is the Kronecker symbol. The system of equations above allows to describe the displacement $\mathbf{u}$ of an elastic body, homogeneous and isotropic, that occupies in its equilibrium position the region Ω, under the action of an external body force, whose density is $\mathbf{f}$ and a surface charge distributed on Γ_N with intensity $\mathbf{g}$. Also sketch a discretization technique based on finite element method in order to get a matrix formulation of the problem.

12. Find the weak formulation of the problem by applying the Green formula:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\Omega \subset \mathbb{R}^2$ with regular boundary $\partial\Omega$, $\Delta^2\cdot$ is $\Delta\Delta\cdot$ the bilaplacian operator, $f \in L^2(\Omega)$ is a given function. Introduce a proper functional space, its norm and semi-norm, check coercivity and continuity of the operator, demonstrate existence and uniqueness for the solution. Introduce also a feasible discretization scheme.

13. Let A be an $n \times n$ non-singular matrix with real eigenvalues and consider the iteration:

$$x^{k+1} = x^k + \alpha(b - Ax^k), \qquad k = 0, 1, 2, \ldots$$

1. Assume that A has both positive and negative eigenvalues. Show that for any given $\alpha \neq 0$, there exists at least one initial vector x^0 such that this iteration diverges.

2. Assume that A has only positive eigenvalues. Derive conditions on α under which the iteration will converge for any x^0. Show how to choose α so that the spectral radius of $I - \alpha A$ will be the smallest.

14. We want to compute the zero α of the function $f(x) = x^3 - 2$ with the fixed point method $x^{(k+1)} = \phi(x^{(k)})$ given as:

$$x^{(k+1)} = x^{(k)}\left(1 - \frac{\omega}{3}\right) + (x^{(k)})^3(1-\omega) + \frac{2\omega}{3(x^{(k)})^2} + 2(\omega - 1), \quad k \geq 0,$$

where $\omega \in \mathbb{R}$ is a real parameter.

(a) For which values of the parameter ω the root of the function f is a fixed point?

(b) For which ω the method is at least of order 2?

(c) Is there a value for ω such that the order of the fixed point is greater than 2?

15. Let us consider the problem: find $y \colon [0, T) \mapsto \mathbb{R}$ such that

$$\begin{cases} y'(t) = \lambda y(t) \\ y(0) = y_0. \end{cases} \tag{7}$$

(a) Introduce a discretization for problem (7) with finite differences, using a forward Euler scheme.

(b) Denote by h the time step of the discretized form of (7). Indicate the absolute stability condition over h.

(c) Let us introduce a perturbed system:

$$\begin{cases} z'(t) = \lambda z(t) + \delta(t) \\ z(0) = y_0 + \delta_0. \end{cases} \tag{8}$$

Let us indicate with y_n and z_n the numerical solutions of (7) and (8), respectively, obtained with forward Euler method. Write the system representing the difference w_n between the perturbed and unperturbed numerical solution. It is possible to see that the following estimate holds:

$$|y_n - z_n| \leq C\epsilon \qquad \forall n \qquad \text{with } |\delta(t)| \leq \epsilon$$

assuming that the time step h is under the stability condition previously defined.

Continuum mechanics

16. An eulerian tensor field A is objective if it transforms under an observer change according to

$$A^*(x^*, t) = Q(t)A(x, t)Q^T(t)$$

where $x^*(p, t) = q(t) + Q(t)(x(p, t,) - O)$ is the rigid motion describing the change of observer. Here O is the origin, $q(t)$ a point in space, and $Q(t)$ a rotation for all time t. Assume that A is smooth and objective, and denote by $L = \operatorname{grad} \mathrm{v}$ the gradient of the eulerian velocity and by W its skew-symmetric part. Which of the following time rates is objective?

(a) $\dot{A}$

(b) $\dot{A} + L^T A + AL$

(c) $\dot{A} - WA + AW$

17. Consider a homogeneous isotropic elastic body. Starting from its natural configuration, the body is loaded by a slowly increasing uniform positive pressure. Assuming that the deformations are small and that linear elasticity theory is applicable, show that its volume will decrease.

18. Prove Archimede's principle stating that a body immersed in fluid at rest is subjected to an upward force equal to the weight of the fluid it displaces. Is this still true if the body is only partially submerged?

19. The time rate of the linear momentum of a continuous body in motion can be calculated as the product of the mass of the body times the acceleration of its centre of gravity. Prove this assertion.

20. Cauchy's stress tensor is one of the key constructs in the classical mechanics of continuous media. Under suitable hypotheses, it is a symmetric tensor.

(a) Discuss the physical meaning of this symmetry property.

(b) Give a proof of this symmetry property.