SISSA – Mathematics Area

Entrance examination for the course in Mathematical Analysis, Modelling, and Applications

May 2, 2017

Solve FIVE of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

Measure theory

1. Let f_k, $f\colon [0,1] \to \mathbb{R}$ be measurable functions and let g_k, $g\colon [0,1] \to \mathbb{R}$ be defined by

$$g_k(x) = f_k(x^2) \quad \text{and} \quad g(x) = f(x^2),$$

for any $x \in [0,1]$.

(a) Prove that, if $f_k \to f$ in $L^p([0,1])$ for some $p > 2$, then $g_k \to g$ in $L^1([0,1])$.

(b) Consider the functions

$$f_k(x) := \begin{cases} \dfrac{1}{\sqrt{x}\log x} & \text{if } 0 < x < \frac{1}{2k}, \\ 0 & \text{if } x = 0 \text{ or } \frac{1}{2k} \le x < 1; \end{cases}$$

show that $f_k \to 0$ in $L^2([0,1])$ and that $g_k \notin L^1([0,1])$.

2. Let $\{f_n\} \subset L^1([0,1])$ be any sequence of measurable functions such that

(a) $f_n \to 0$ almost everywhere;

(b) $\sup_n \displaystyle\int_0^1 (f_n^-)^2 < \infty$, where g^- denotes the negative part of any function, $g^-(x) = -\min\{0, g(x)\}$;

(c) and $\displaystyle\int_0^1 f_n \to 0$.

Prove that $f_n \to 0$ in $L^1([0,1])$. Hint: show first that $f_n^- \to 0$ in $L^1([0,1])$ and then use the identity $|g| = g + 2g^-$, valid for any function g.

3. Let $f\colon [0,1] \to \mathbb{R}$ be any measurable function and for any $t > 0$ define $A_t := \{x \in [0,1]\colon |f(x)| > t\}$.

(a) Prove that

$$\limsup_{p\to+\infty} \frac{\int_0^1 |f|^{p+1}}{\int_0^1 |f|^p} \le \operatorname*{ess\,sup}_{x\in[0,1]} |f(x)| \ ;$$

(b) Prove that for each $t > 0$ such that $|A_t| > 0$, i.e., the Lebesgue measure of A_t is strictly positive, it holds true

$$\liminf_{p\to+\infty} \int_{A_t} \left|\frac{f(x)}{t}\right|^p dx = +\infty \ ;$$

(c) Prove that for each $t > 0$ such that $|A_t| > 0$ it holds true

$$\liminf_{p\to+\infty} \frac{\int_0^1 |f|^{p+1}}{\int_0^1 |f|^p} \ge t.$$

Differential equations

4. Solve the following Cauchy problem

$$\begin{cases} y'(x) = \dfrac{(y(x))^2}{1-(y(x))^2}, \\ y(0) = 1/2, \end{cases}$$

and compute the limits

$$\lim_{x\to 1/2-} y(x) \quad \text{and} \quad \lim_{x\to 1/2-} y'(x).$$

5. Let $\omega\colon [0,+\infty) \to [0,\infty)$ be a continuous function such that $\omega(0) = 0$ and $\omega(y) > 0$ for every $y > 0$, and let $y\colon [0,+\infty) \to [0,+\infty)$ a C^1 function such that

$$\begin{cases} |y'(x)| \le \omega(y(x)) \quad \forall x \in [0,+\infty), \\ y(0) = 0. \end{cases}$$

Prove that:

(a) If

$$\int_0^1 \frac{dy}{\omega(y)} = +\infty,$$

then $y(x) = 0$, for all $x \in [0,+\infty)$.

(b) If

$$\omega(y) \le \frac{y^{1-\alpha}}{\alpha} \qquad \forall y \in (0,1) \quad \forall \alpha \in (0,1),$$

then $y(x) = 0$ for each $x \in [0,+\infty)$. (Hint: for $y \in (0,1)$ calculate $\inf_{\alpha\in(0,1)} \frac{y^{1-\alpha}}{\alpha}$).

6. Find a $C^1((0,\infty)^2) \cap C^0([0,\infty)^2)$ solution u of the partial differential equation

$$\begin{cases} \partial_x u(x,y) + \partial_y u(x,y) = u(x,y), \quad x, y > 0, \\ u(0,y) = \sin(y), \\ u(x,0) = -\sin(x). \end{cases}$$

Prove that such a solution is unique in $C^1((0,\infty)^2) \cap C^0([0,\infty)^2)$.

Functional analysis

7. For any integer number $k \geq 1$ let $T_k \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the linear and continuous operator defined by

$$(T_k u)(x) = u(x + \tfrac{1}{k}).$$

Let I be the identity operator over $L^2(\mathbb{R})$ and let $\mathcal{L}(L^2(\mathbb{R}), L^2(\mathbb{R}))$ be the Banach space of linear and continuous operators from $L^2(\mathbb{R})$ into itself endowed with the norm

$$\|T\|_{\mathcal{L}} := \sup\{\|Tu\|_{L^2} : u \in L^2(\mathbb{R}),\ \|u\|_{L^2} \leq 1\}.$$

(a) Prove that for each $u \in L^2(\mathbb{R})$ it holds true $T_k u \to u$, strongly in $L^2(\mathbb{R})$.

(b) Prove that the sequence of the norms $\|T_k - I\|_{\mathcal{L}}$ does not converge to 0 as $k \to +\infty$.

8. Consider the operator

$$T : C([0,1]) \to C([0,1]), \qquad (Tu)(x) = \int_0^{x^2} u(\sqrt{s})\, ds.$$

Prove that for each $\lambda \in \mathbb{R}$ it does not exist $u \in C([0,1])$, different from the null function, solution of $Tu = \lambda u$.

9. Let (X, d) be a complete metric space and let $F : X \to X$. Define the n-th iteration of F as follows:

$$F_n(x) := \underbrace{(F \circ F \circ \cdots \circ F)}_{n\text{-times}}(x),$$

and assume for each $n \in \mathbb{N}$ the existence of $\alpha_n \geq 0$ such that

$$d(F_n(x), F_n(y)) \leq \alpha_n d(x,y), \qquad \forall x, y \in X.$$

Prove that, if $\sum_n \alpha_n < +\infty$, then there exists a unique $\bar{x} \in X$ such that $F(\bar{x}) = \bar{x}$.

10. Consider the following space $X = \{u \in C^1([-1,1])\colon u(-1) = 0, u(1) = 1\}$, and the functional $I : X \to [0,\infty)$ defined as follows:

$$I(u) := \int_{-1}^{1} (u'(x))^2(1-u'(x))^2 dx\,;$$

prove that $\inf_{u\in X} I(u) = 0$ and that such inf is not attained inside X.

Numerical analysis

11. Consider the linear system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -\alpha^2 & 1 & 0 \\ 0 & -4\alpha & 1 \end{bmatrix},$$

$\mathbf{b}$ is such that $\mathbf{x} = [1,1,1]^T$ is the solution to $A\mathbf{x} = \mathbf{b}$, and $\alpha \in \mathbb{R}$.

(a) Define the Jacobi method for the numerical approximation of the solution of the linear system $A\mathbf{x} = \mathbf{b}$. Define and compute its iteration matrix.

(b) Define the Gauss-Seidel method for the numerical approximation of the solution of the linear system $A\mathbf{x} = \mathbf{b}$. Define and compute its iteration matrix.

(c) Discuss for which values of α both Jacobi and Gauss-Seidel methods converge. Which one is converging faster? State (without proof) any convergence result that you use.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f'(x) > 0$, and denote by α one of its roots. The fixed point method

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2\left[f'(x_n)\right]^2 - f(x_n)f''(x_n)}$$

for solving the nonlinear equation $f(x) = 0$ is known as Halley's method.

(a) Let $g(x) = \frac{f(x)}{\sqrt{f'(x)}}$. Show that Halley's method is equivalent to Newton's method applied to $g(x)$.

(b) Let α be a simple root of f. Show that the convergence of Halley's method is at least cubic. Compare to the order of convergence of Newton's method.

13. What is the maximum degree of exactness that the following quadrature formula

$$\int_0^1 f(x)\frac{1}{\sqrt{x}}\,\mathrm{d}x \approx w_0 f(x_0) + w_1 f(x_1)$$

can attain? Provide all relevant definitions and theorems (without proof). Compute weights w_0, w_1 and nodes x_0, x_1 so that the maximum degree of exactness is reached.

14. The *modified Euler method* for the initial value problem

$$\begin{cases} y'(t) = f(t, y), & t \in [0, T) \\ y(0) = y_0 \end{cases}$$

is given by the following formula

$$y_{n+1} = y_n + hf\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(t_n, y_n)\right),$$

where $y_n \approx y(t_n)$ and $t_{n+1} = t_n + h$.

(a) Provide a direct proof that this method is convergent. What is the order of convergence?

(b) This method is a particular instance of a class of explicit methods. Provide their name, their general formulation, and a summary (without proof) of theoretical results concerning consistency, stability and convergence.

(c) Denote by p the order of convergence of the modified Euler method, as found at question (a). Is this the only method among the ones in question (b) that has order of convergence equal to p and requires two evaluations of f? If not, list other methods that satisfy these assumptions.

15. Consider the advection diffusion problem: find $u : [0, 1] \to \mathbb{R}$ such that:

$$\begin{cases} -(\mu u')' + bu' = 0, & \text{for } x \in (0, 1), \\ u(0) = 0, \\ u(1) = 1, \end{cases}$$

where $\mu = \mu(x)$ and $b = b(x)$.

(a) Write the weak formulation of the problem, by specifying the functional spaces for trial and test functions. Note any additional assumptions on μ and b.

(b) Discuss the well posedness of the problem. State (withouth proof) all theorems that you use. Note any additional assumptions on μ and b.

(c) Suppose that the problem is "convection dominated", i.e. $\|\mu\|/\|b\|$ is very small. Illustrate possible stabilization methods.

Continuum mechanics

16. The Navier-Stokes equations result from

(a) the balance of linear momentum,

(b) a constitutive equation expressing the Cauchy stress as a function of the velocity gradient.

Explain why the Cauchy stress cannot depend on the skew-symmetric part of the velocity gradient, and sketch the derivation of the Navier-Stokes equation starting from (a) and (b).

17. State and prove the equation establishing the balance of linear momentum for a control volume (fixed in space).

18. The Cauchy stress tensor and the stretching tensor (the symmetric part of the deformation gradient) are a power-conjugate pair, in the sense that their product defines a power density.

(a) Show that the first Piola-Kirchhoff stress is power conjugate to the time derivative of the deformation gradient.

(b) Can you exhibit any other power conjugate stress/rate-of-deformation pair (and prove that the elements of the pair are indeed power conjugate)?

19. A *simple shear* is a deformation $\mathbf{y} = \mathbf{y}(\mathbf{x})$ such that (in cartesian components)

$$\begin{aligned} y_1 &= x_1 + \gamma\, x_2\,, \\ y_2 &= x_2\,, \\ y_3 &= x_3\,, \end{aligned}$$

where γ is a scalar. Compute the principal stretches assuming that $\gamma > 0$.

20. Consider a thin, cylindrical capillary tube of constant radius $r = 100\,\mu\text{m}$ and thickness $t = 10\,\mu\text{m}$, subject to the internal pressure p. Let $\sigma_\text{f} = 40\,\text{MPa}$ be the strength (breaking stress) of glass. Estimate the pressure p_f at which the glass capillary breaks (for simplicity, assume the capillary of infinite length).