SISSA – Mathematics Area

Entrance examination for the course in Mathematical Analysis, Modelling, and Applications

April 12, 2018

Solve FIVE of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

Mathematical Analysis

1. Let $f\colon L^1([0,2])$, let $\psi\colon [0,1] \to [0,1]$ be a function, and let $F\colon [0,1] \to \mathbb{R}$ be defined by

$$F(t) = \int_0^1 f(x + \psi(t))dx \quad \text{for every } t \in [0,1].$$

(a) Prove that, if ψ is continuous, then F is continuous.

(b) Prove that, if $\psi \in C^1([0,1])$ and $\psi'(t) > 0$ for every $t \in [0,1]$, then F is differentiable for almost every $t \in [0,1]$.

2. Let (f_n) be a bounded sequence in $L^1([0,1])$.

(a) Prove that

$$\liminf_{n\to\infty} f_n(x) < +\infty \quad \text{for almost every } x \in [0,1]\,.$$

(b) Find an example of a bounded sequence (g_n) in $L^1([0,1])$ such that

$$\limsup_{n\to\infty} g_n(x) = +\infty \quad \text{for almost every } x \in [0,1]\,.$$

3. Consider the differential equation

$$\begin{cases} y''(x) = (y'(x))^2 - 2\,, \\ y(0) = 0\,, \\ y'(0) = 1\,. \end{cases}$$

Prove that

(a) the solution is defined for all $x \in \mathbb{R}$,

(b) $\lim_{x \to +\infty} y'(x) = -\sqrt{2}$,

(c) $\lim_{x \to +\infty} y(x) = -\infty$.

4. Let y be the maximal solution of the Cauchy problem

$$\begin{cases} y''(x) = -y(x)^3 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

(a) Prove that

$$\frac{1}{2}(y'(x))^2 + \frac{1}{4}(y(x))^4 = \frac{1}{4}$$

for every $x \in \mathbb{R}$ for which $y(x)$ is defined. Prove that

(b) $y(x)$ is defined for every $x \in \mathbb{R}$,

(c) $y(-x) = y(x)$ for every $x \in \mathbb{R}$,

(d) the function y is periodic.

5. Let $A : \ell^2(\mathbb{Z}; \mathbb{C}) \mapsto \ell^2(\mathbb{Z}; \mathbb{C})$ be a linear operator acting on the space

$$\ell^2(\mathbb{Z}; \mathbb{C}) := \left\{ (u_i)_{i \in \mathbb{Z}}\,,\ u_i \in \mathbb{C}\,,\ \|u\|_2^2 := \sum_{i \in \mathbb{Z}} |u_i|^2 < +\infty \right\}.$$

Let A^i_j be the matrix elements defined by

$$(Au)_j := \sum_{i \in \mathbb{Z}} A^i_j u_i\,, \quad \forall j \in \mathbb{Z}\,.$$

The operatorial norm of A is defined by $\|A\| := \sup_{\|u\|_2 \le 1} \|Au\|_2$. Prove that

(a) $\sup_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} |A^i_j|^2 \le \|A\|^2$,

(b) the linear operator B defined by the matrix elements

$$B^i_j := \frac{|A^i_j|}{|i - j|}\,,\ \forall i \ne j\,, \quad B^i_i := 0\,,$$

is bounded and $\|B\| \le C\|A\|$ for a suitable constant $C > 0$ independent of $\|A\|$.

6. Let $A : H \to H$ be a linear operator *everywhere* defined on a Hilbert space H. Assume that A is symmetric, namely

$$(Ax, y) = (x, Ay) \quad \text{for every } x, y \in H\,,$$

where $(\ ,\ )$ denotes the scalar product on H. Prove that A is bounded.

7. Let $f\colon \mathbb{R}^+ \to \mathbb{R}$ be a continuous functions. Assume that

$$\lim_{n\to\infty,\ n\in\mathbb{N}} f(nx) = 0 \qquad \text{for all } x > 0.$$

(a) Show that

$$\lim_{x\to\infty} f(x) = 0.$$

(b) Prove that the statement of point (a) above is not true if we assume only that

$$\lim_{n\to\infty,\ n\in\mathbb{N}} f(2^n x) = 0 \qquad \text{for all } x > 0.$$

8. Let $u_0, u_1\colon \mathbb{R} \to \mathbb{R}$ be periodic functions of period 2π and of C^∞ regularity. Using the Fourier series in x, exhibit a solution (periodic in x of period 2π) to the Cauchy problem

$$\partial_t^2 u + \partial_t u + \partial_x u = 0, \quad t > 0,\ x \in \mathbb{R},$$
$$u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}.$$

9. Let m be a Borel probability measure on $[0,1]$, and let $m \otimes m$ be the product probability measure on $[0,1]^2$.

(a) Prove that there are points $x_n \in [0,1]$, $n \in \mathbb{N}$, and a Borel measure m^c on $[0,1]$, such that

$$m = \sum_{n\in\mathbb{N}} c_n \delta_{x_n} + m^c,$$

where $c_n := m(\{x_n\})$, δ_x is the Dirac measure at x, and $m^c(\{x\}) = 0$ for all $x \in [0,1]$.

(b) Let $D = \{(x,y) : x = y\} \subset [0,1]^2$. Show that

$$m \otimes m(D) = \sum_n c_n^2.$$

(c) Deduce that

$$\max\big\{m(\{x\}), x \in [0,1]\big\} \geq m \otimes m\big(\{(x,y) : x = y\}\big).$$

10. Consider the set

$$K = \left\{ f \in C^1([0,1],\mathbb{R}) : f(0) = 0, |f'(x)| \leq 1 \right\} \subset C^0([0,1],\mathbb{R}).$$

(a) Show that K is pre-compact in $C^0([0,1])$.

(b) Prove that for all $n \in \mathbb{N}$, one can cover K with 4^n closed balls in $C^0([0,1])$ of radius $1/n$ (Hint: Consider balls centred at suitable piecewise affine functions with slope ± 1).

Numerical Analysis

11. To avoid computing e^x repeatedly, we construct an array of equi-spaced values of e^x and then, for a given value of x, use linear interpolation with the nearest array values to efficiently obtain an approximation for e^x.

(a) Give a derivation of the error formula for linear interpolation.

(b) Use your error formula to estimate the fewest number of equispaced values in $[0,1]$ that are required to insure that the error in the approximate value for e^x is less than $1/2 \times 10^{-6}$ for any $x \in [0,1]$. Justify your estimate.

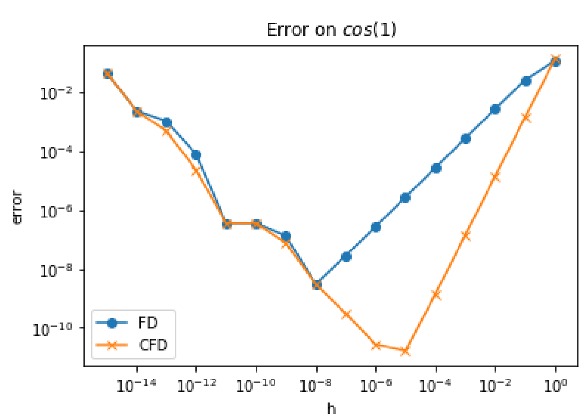
12. Consider the following two algorithms to compute numerically the derivative of a function:

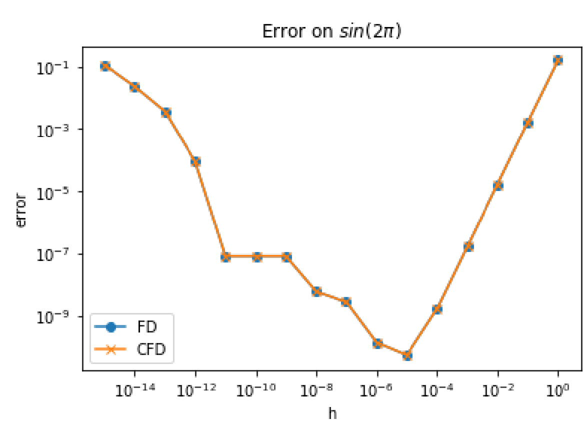
(I) $FD(f;x;h) := \frac{f(x+h)-f(x)}{h}$

(II) $CFD(f;x;h) := \frac{f(x+h)-f(x-h)}{2h}$

We apply the two algorithms to the functions $\cos(x)$ and $\sin(x)$, evaluated respectively at the points $x=1$, and $x = 2\pi$.

We plot the values of $e_{FD} := |FD(\cos;1;h) + \sin(1)|$ and $e_{CFD} := |CFD(\sin;1;h) - \cos(2\pi)|$ for different values of h and obtain:





(a) Explain the difference you see in the inclination of the error plots on the left when $h \in [10^{-5}, 10^{0}]$.

(b) Motivate why the error plots (both on the left and on the right plots) decrease up to a certain value of h, and then start raising again.

(c) Motivate why the error plots on the right differ from the error plots on the left, by providing an estimate of the truncation error in the two considered cases.

13. Consider the linear system $Ax = b$, with $x, b \in \mathbb{R}^n$ and $A = M - N \in \mathbb{R}^{n\times n}$. Assume that M is nonsingular and that $(M^{-1}N)^k \to 0$ as $k \to \infty$.

(a) show that the iterates x_k defined by

$$Mx_{k+1} = Nx_{x_k} + b,$$

converge to $x = A^{-1}b$ for any starting vector x_0.

(b) Find a splitting $A = M - N$ for the matrix $A = \begin{pmatrix} 10 & -1 \\ -1 & 10 \end{pmatrix}$, so that the iteration in (a) is convergent. Justify your answer.

14. Let A be a $n \times n$ singular symmetric matrix with eigenvalues $\lambda_1 = 0$ and $\lambda_i > 0$ for $i = 2, 3, \ldots, n$. Under what conditions on the factor α, the vector b and the initial iterate x^0, will the iteration

$$x^* = b - (A - I)x^n$$
$$x^{n+1} = \alpha x^* + (1 - \alpha)x^n$$

converge to a solution of $Ax = b$? Justify your answer.

15. Assume that $f(x) : \mathbb{R} \mapsto \mathbb{R}$ is a smooth function. Let $\epsilon > 0$ and consider the three data values $(0, f(0))$, $(\epsilon, f(\epsilon))$, and $(1, f(1))$. Let $p(x)$ be the polynomial that arises as the limit of the polynomial interpolant of the data as $\epsilon \to 0$.

(a) What is the degree of $p(x)$?

(b) What data (if any) does $p(x)$ interpolate?

(c) What data (if any) does $p'(x)$ interpolate?

Continuum Mechanics

16. Consider the two-dimensional, unsteady flow of an incompressible fluid whose velocity field is given by $\mathbf{v}(x_1, x_2, t) = \alpha\, x_1 x_2\, \mathbf{e}_1 + v_2(x_2, t)\mathbf{e}_2$, where $\alpha \in \mathbb{R}$, $\mathbf{e}_1$-$\mathbf{e}_2$ is an orthonormal basis in $\mathbb{R}^2$, $t > 0$ is the time, and x_i, $i = 1, 2, 3$, are the space variables. The (homogeneous) density field is $\rho(x_1, x_2) = \rho_0 > 0$. Determine the velocity component $v_2(x_2, t)$ such that $v_2(1, t) = \beta \exp(-t)$, $\beta \in \mathbb{R}$. Let $\mathcal{B} = [0, 1] \times [0, 1] \times [0, 1] \in \mathbb{R}^3$ be a fixed control volume. Compute the (instantaneous) time rate of the total momentum of the fluid particles that are contained in $\mathcal{B}$ at $t = \bar{t}$.

17. Consider a steady and irrotational flow with velocity field $\mathbf{v}$, density ρ, and Cauchy stress $\mathbf{T}$ a pressure, that is $\mathbf{T} = -\pi\mathbf{I}$ with π a scalar field. Derive Bernoulli's theorem by assuming that the body force $\mathbf{b}$ is conservative with potential β.

18. A cylindrical chalk stick with circular cross section of radius $r = 5\,\mathrm{mm}$ is subject at its extremities to applied torques of magnitude T. Let $\sigma_{\mathrm{f}} = 1\,\mathrm{MPa}$ be the strength of chalk. Estimate the torque T_{f} at which the chalk stick breaks.

19. An elastic cylindrical body is subject on its bases to applied tractions (per unit reference area) $\mathbf{Sn} = \sigma\mathbf{n}$, where $\mathbf{S}$ is the first Piola-Kirchhoff stress tensor, $\mathbf{n} = \pm\mathbf{e}_1$ is the outward unit normal to the bases of the cylinder, and $\mathbf{e}_1$ is the unit vector along its axis. Assuming a neo-Hookean incompressible constitutive law, establish the relation between the loading parameter σ and the principal stretch along the cylinder axis (for an incompressible neo-Hookean material the Cauchy stress $\mathbf{T} = -\pi\mathbf{I} + \mu\mathbf{F}\mathbf{F}^{\mathrm{T}}$, with μ the shear modulus of the material, π the Lagrange multiplier associated with the incompressibility constraint, and $\mathbf{F}$ the deformation gradient).

20. Let $\mathbf{RU} = \mathbf{F}$ and $\mathbf{VR} = \mathbf{F}$ be the *right* and *left* polar decompositions of the deformation gradient $\mathbf{F}$, respectively. Derive the transformation laws for $\mathbf{F}$, $\mathbf{R}$, $\mathbf{U}$, and $\mathbf{V}$ due to a change in observer.