SISSA – Mathematics Area

Entrance examination for the course in Mathematical Analysis, Modelling, and Applications

September 5, 2018

Solve FIVE of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

Mathematical Analysis

1. Let f_n be an approximate identity in $L^1(\mathbb{R})$:

- $f_n \geq 0$;
- $\int f_n = 1$;
- $\lim_{n\to\infty} \int_{|t|\geq\delta} f_n(t)dt = 0$.

Show that $\lim_{n\to\infty} \|f_n\|_{L^p} = \infty$, for all $p > 1$.

2. Let $A \subset \mathbb{R}$ be a measurable set with $|A| > 0$, i.e. of positive Lebesgue measure. Prove that for any $n \in \mathbb{N}$, A contains arithmetic progressions of length n, that is there exist $\varepsilon > 0$ and $x_1, \ldots, x_n \in A$ such that $x_i - x_{i+1} = \varepsilon$ for $i = 1, \ldots, n-1$.

3. Let $y = y(x)$ be the unique solution of the Cauchy problem

$$\begin{cases} y' = \ln(\sqrt{1+y^2}), \\ y(0) = y_0. \end{cases}$$

(a) Prove that if $y_0 > 0$, then y is strictly increasing and convex; if $y_0 < 0$, then y is strictly increasing and concave.

(b) Prove that y is globally defined.

4. Consider the PDE

$$\partial_t u + u\partial_x u = u, \qquad u(0,x) = u_0(x) = -\tanh(x).$$

Let $X(t,y,z), U(t,y,z)$ be the solutions to ODE Cauchy problem

$$\frac{dX}{dt} = U(t), \qquad \frac{dU}{dt} = U(t), \quad X(0,y,z) = y,\ U(0,y,z) = z.$$

(a) Show that the function implicitly given by

$$u(t, X(t,y,u_0(y))) = U(t,y,u_0(y)),$$

is a C^1 solution to the PDE for a small time.
[Hint: Use Implicit Function Theorem]

(b) Estimate the maximal time for which the solution given by Point (a) can be computed.

5. Prove that a continuous periodic function with two incommensurable periods is a constant. Prove that differential equation $\dot{x} + a(t)x = b(t)$ cannot have three non-constant periodic solutions $x_1(t), x_2(t), x_3(t)$ with mutually incommensurable periods.

6. Let (X, d) a compact metric space and $f : X \to X$ an isometry, i.e.

$$d(x, y) = d(f(x), f(y)), \qquad \forall\, x, y \in X.$$

Prove that f is surjective. Prove that compactness of (X, d) is necessary.

7. Let $f_n(x) = \sum_{k=0}^{2n} \frac{x^k}{k!}$. Prove that the equation $f_n(x) = 0$ has no real roots.
[Hint: Use Induction.]

8. Let H be an Hilbert space with scalar product $(\cdot, \cdot)$ and $T : H \to H$ be a linear operator of the form

$$Tx = \sum_n (x, a_n) b_n,$$

where $a_n, b_n \in H$ and

$$\sum_n |a_n||b_n| < \infty.$$

Prove that if $x_n \in H$ is a weakly convergent sequence, then Tx_n is strongly convergent.

9. (a) Let $L : \mathbb{R} \to \mathbb{R}$ be strictly convex and assume that the function $u : \mathbb{R} \to \mathbb{R}$ satisfies

$$L(u(x)) + L(u(y)) \leq L\Big(\frac{u(x) + u(y)}{2} + \frac{x - y}{2}\Big) + L\Big(\frac{u(x) + u(y)}{2} - \frac{x - y}{2}\Big),$$

for all $x, y \in \mathbb{R}$. Prove that the function u is Lipschitz.
[Hint: Use strict convexity about the point $(u(x) + u(y))/2$.]
(b) Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be C^2 and uniformly convex, i.e.

$$\frac{1}{C}|v|^2 \leq v^T \nabla^2 L(x) v \leq C|v|^2.$$

Assume that the function $u : \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$L(u(x)) + L(u(y)) \leq L\Big(\frac{u(x) + u(y)}{2} + \frac{x - y}{2}\Big) + L\Big(\frac{u(x) + u(y)}{2} - \frac{x - y}{2}\Big),$$

for all $x, y \in \mathbb{R}$. Prove that the function u is Lipschitz.
[Hint: Use the Taylor expansion about the point $(u(x) + u(y))/2$.]

10. Let $E \subset C[0, 1]$ be a finite dimensional subspace and $f_n \in E$, $n = 1, 2, \dots,$ a sequence such that $\|f_n\| \to \infty$ as $n \to \infty$. Prove that there exists a segment $\Delta \subset [0, 1]$ and a subsequence f_{n_m}, $m = 1, 2, \dots,$ such that $\min_{x \in \Delta} |f_{n_m}(x)| \to \infty$ as $m \to \infty$.

Numerical Analysis

11. Consider the following initial value problem

$$\dot{\boldsymbol{u}} = \boldsymbol{f}(t, \boldsymbol{u}), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0,$$

with $t \in [0, T]$ and $T > 0$.

(a) explain the steps necessary to implement an algorithm that solves the problem through a time integration method. Describe the characteristics of the method you choose.

Assume $\boldsymbol{f}$ to be a nonlinear function in both its arguments. Illustrate the following two methods to solve the nonlinear system of equations arising from the time discretization in (a):

(b) the fixed-point iteration method,

(c) the Newton method.

12. Consider the following differential equation

$$-u'' + qu' = 1 \text{ in } [0, \pi] \tag{1}$$

for some $q \in \mathbb{R}$.

(a) Derive the weak formulation of the differential equation (1) with boundary conditions $u(0) = u(\pi) = 0$. Specify the function space V in which you expect to find the weak solution u.

(b) Show that the bilinear form $a(\cdot, \cdot)$ introduced in (a) is coercive, i.e. there exists $\alpha > 0$ such that $a(v, v) \geq \alpha \|v\|_V^2$ for all $v \in V$.

(c) Derive the weak formulation of the differential equation (1) with boundary conditions $u(0) = 0, u'(\pi) = 0$. Again, write down the corresponding function space W.

(d) Show that there exists $q \in \mathbb{R}$ and $v \in W$ such that the bilinear form introduced in (c) is not coercive, by providing a counterexample for a value of q and an explicit expression for v.

(e) Let $m \in \mathbb{N}$ be a fixed integer. Divide the domain $[0, \pi]$ into m equi-spaced intervals and introduce a finite element discretization of the problem (1).

(f) In case (a), do you expect the finite element discretization introduced in (e) to be accurate for any q? If not, how would you improve the accuracy of the discretization (while keeping m fixed)?

(g) In case (c) and assuming $q > 0$, do you expect the finite element discretization introduced in (e) to be accurate for any q? If not, how would you improve the accuracy of the discretization (while keeping m fixed)?

13. Consider the linear system $\mathbb{A}\mathbf{x} = \mathbf{b}$, where $\mathbb{A} \in \mathbb{R}^{n \times n}$ is a positive definite matrix and $\mathbf{b} = [1, \ldots, 1]^T \in \mathbb{R}^n$. Moreover, let $\mathbb{H}$ and $\mathbb{S}$ be defined as

$$\mathbb{H} = \frac{\mathbb{A} + \mathbb{A}^T}{2}, \quad \mathbb{S} = \frac{\mathbb{A} - \mathbb{A}^T}{2}.$$

(a) Which theoretical results can you use to localize the eigenvalues of $\mathbb{S}$ on the complex plane? How would you solve the linear system $(\alpha\mathbb{I} + \mathbb{S})\mathbf{y} = \mathbf{s}$ for a fixed $\alpha > 0$ and multiple random vectors $\mathbf{s}$?

(b) Which numerical methods can you use to estimate the minimum and maximum eigenvalues of $\mathbb{H}$? How would you solve the linear system $(\alpha\mathbb{I}+\mathbb{H})\mathbf{z} = \mathbf{h}$ for a fixed $\alpha > 0$ and multiple random vectors $\mathbf{h}$?

Let $\alpha > 0$, consider the following iterative method: given an initial guess $\mathbf{x}^{(0)}$ compute

$$\begin{cases} (\alpha\mathbb{I} + \mathbb{H})\mathbf{x}^{(k+1/2)} = (\alpha\mathbb{I} - \mathbb{S})\mathbf{x}^{(k)} + \mathbf{b}, \\ (\alpha\mathbb{I} + \mathbb{S})\mathbf{x}^{(k+1)} = (\alpha\mathbb{I} - \mathbb{H})\mathbf{x}^{(k+1/2)} + \mathbf{b}. \end{cases} \tag{2}$$

(a) Is the method (2) consistent for any $\alpha > 0$?

(b) For a fixed $\alpha > 0$, how can you check *a priori* if the method (2) is convergent?

14. Consider the following quadrature rule

$$Q(f) = w_1 f(x_1)$$

for the computation of the weighted integral

$$W(f) = \int_0^1 x^\alpha f(x)\,\mathrm{d}x, \quad \alpha > 0.$$

(a) For any $\alpha > 0$, determine w_1 and x_1 so that the proposed formula is exact for $f(x) = 1$ and $f(x) = x$.

(b) Introduce the notion of degree of exactness of a quadrature rule for a standard (non-weighted) integral

$$I(g) = \int_0^1 g(x)\,\mathrm{d}x,$$

and note how $W(f(x)) = I(x^\alpha f(x))$. Assuming $\alpha \in \mathbb{N}$, by letting $\alpha \to \infty$, can you exploit the result at (a) to obtain a quadrature rule for the computation of I which is characterised by arbitrarily large degree of exactness? Motivate the answer and compare to theoretical results.

(c) For $\alpha = 1/2$ and under the assumption $f \in \mathcal{C}^2([0,1])$, provide an estimate of the integration error.

15. Consider the following transport diffusion equation with $\mu > 0$ and $\mathrm{div}(\boldsymbol{u}) = 0$ with homogeneous Dirichlet boundary condition and $\boldsymbol{s} \neq \mathbf{0}$ a generic given forcing function

$$\dot{\boldsymbol{u}} - \mathrm{div}(\mu\nabla\boldsymbol{u}) + \mathrm{div}(\boldsymbol{u}\otimes\boldsymbol{u}) = \boldsymbol{s}.$$

Choose only one of the two exercises:

Option 1

(a) Write the equation in integral form and making use of the Gauss theorem transform it in therms of fluxes across the surface of the control volume. Starting from the integral form of the equation describe the steps necessary to implement a finite volume discretization.

(b) Why is the convective term $\text{div}(\boldsymbol{u} \otimes \boldsymbol{u})$ written in this form? Is it always true that $\text{div}(\boldsymbol{u} \otimes \boldsymbol{u}) = \boldsymbol{u} \cdot \nabla \boldsymbol{u}$?

(c) Describe how the geometrical properties of the finite volume mesh affect the accuracy of the resulting method. For a fixed number of cells, discuss the mesh generation algorithm that would give the best results. If the mesh was generated instead through a different algorithm, what numerical techniques can be employed to improve the accuracy?

Option 2

(a) Write all the steps necessary to discretize the equation with a spectral element method (hint: start from the weak formulation)

(b) Is it always true that $\text{div}(\boldsymbol{u} \otimes \boldsymbol{u}) = \boldsymbol{u} \cdot \nabla \boldsymbol{u}$?

(c) What are the advantages and disadvantages of a spectral element method respect to a standard finite element method?

Continuum Mechanics

16. Define a rigid deformation and prove that f is a rigid deformation if and only if it admits the representation $f(p) = f(q) + \mathbf{R}(p - q)$, for all $p, q \in \mathcal{B}$, with $\mathbf{R}$ a rotation and $\mathcal{B}$ a body.

17. Show that the Cauchy stress is a symmetric tensor for a continuum subject to body and contact forces.

18. A body $\mathcal{B}$ with density ρ_0 per unit reference (undeformed) volume undergoes a motion $f(X,t)$, with $X \in \mathcal{B}$ a material point and $t > 0$ the time, such that $x_1 = e^t X_1 - e^{-t} X_2$, $x_2 = e^t X_1 + e^{-t} X_2$, $x_3 = X_3$ are the Cartesian coordinates of a point $x = f(X,t)$ in the deformed configuration and X_i, $i = 1, 2, 3$ are those of the corresponding material point. The Cauchy stress tensor associated to such a motion is

$$\mathbf{T} = x_1^2 \mathbf{e}_1 \otimes \mathbf{e}_1 + \alpha x_2 x_3^2 (\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + x_2^2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \beta x_1^3 \mathbf{e}_3 \otimes \mathbf{e}_3,$$

with $\mathbf{e}_i$, $i = 1, 2, 3$ an orthonormal basis in $\mathbb{R}^3$ and $\alpha > 0$, $\beta > 0$ scalar constants. Find the body force $\mathbf{b}$ and contact force $\mathbf{c}$ per unit area such that the balance of linear momentum and the conservation of mass for $\mathcal{B}$ are satisfied. Assume that the contact force $\mathbf{c}$ acts at a point x of the plane tangential to a sphere with radius $x - o$, where o is the origin of $\mathbb{R}^3$.

19. Consider an elastic rod of length l, straight in its unloaded configuration and with circular cross-section of constant radius r. Let the rod be hinged at its extremities and subject to a constant thermal load of magnitude δT. Compute the thermal load δT_b at which the rod buckles (for a solid circular cross-section of radius r the moment of inertia reads $\pi r^4/4$).

20. Consider the two-dimensional, unsteady flow of a fluid whose velocity field is given by $\mathbf{v}(x,y,t) = \sqrt{x^2 + y^2}\, \mathbf{e}_1 + y t^2 \mathbf{e}_2$, where $\mathbf{e}_1$-$\mathbf{e}_2$ is an orthonormal basis in $\mathbb{R}^2$, $t > 0$ is the time, and x, y are the space variables. Given the temperature field $T(x,y,t) = (4x-3)y + A \log(1+t)$, find the constant A such that the instantaneous rate of change of the temperature of the fluid particles crossing the point $x = 1$, $y = 1$ at $t = 0$ equals $\sqrt{2}$. For the value of A just found, compute the time rate of the temperature field in the same point and for the same time.