SISSA – Mathematics Area

Entrance examination for the course in Mathematical Analysis, Modelling, and Applications

March 27, 2019

Solve FIVE of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

Mathematical Analysis

1. Let $P := \{f \in L^2((0,1)) : f \geq 0 \text{ a.e. in } (0,1)\}$. For every $f \in L^2((0,1))$ let $f^+ \in L^2((0,1))$ be the function defined by $f^+(x) := \max\{f(x), 0\}$ for every $x \in [0,1]$. Prove that

(a) for every $f \in L^2((0,1))$ the function f^+ is the projection of f onto the convex set P;

(b) $\|f^+ - g^+\|_{L^2} \leq \|f - g\|_{L^2}$ for every $f,\ g \in L^2((0,1))$.

2. For every function $f\colon (0,1) \to \mathbb{R}$ let $f^+(x) := \max\{f(x), 0\}$ for every $x \in (0,1)$.

(a) Let $f\colon (0,1) \to \mathbb{R}$ be a function and let $x_0 \in (0,1)$. Suppose that $f(x_0) = 0$ and that f and f^+ are differentiable at x_0. Prove that

$$(f^+)'(x_0) = f'(x_0) = 0\,.$$

(b) Using the well known fact that every Lipschitz function is differentiable almost everywhere, prove that, if $g\colon (0,1) \to \mathbb{R}$ is Lipschitz, then $g'(x) = 0$ for almost every $x \in g^{-1}(\{0\})$.

3. Let $y \in C^1(\mathbb{R})$ be a function such that

$$\begin{cases} y'(x) = \sin(y(x) + x^2) & \text{for every } x \in \mathbb{R}\,, \\ y(0) = 0\,. \end{cases}$$

Prove that

(a) $y \in C^\infty(\mathbb{R})$,

(b) $y'(0) = y''(0) = 0$ e $y'''(0) > 0$,

(c) $y(x) > 0$ for every $x \in (0, \sqrt{\pi})$ and $y(x) < 0$ for every $x \in (-\sqrt{\pi}, 0)$.

4. Let $\mathbb{Q}$ be the set of rational numbers and let B be the Banach space of continuous functions from $[0,1]$ to $\mathbb{R}$ equipped with the norm $\|f\| := \sup_{t\in[0,1]} |f(t)|$.

(a) Prove that for every $t \in [0,1]$ and for every continuous functoion $f\colon [0,1] \to \mathbb{R}$ it holds

$$\limsup_{h\to 0} \frac{f(t+h)-f(t)}{h} = \inf_{\delta>0} \sup_{h\in(-\delta,\delta)\cap\mathbb{Q}} \frac{f(t+h)-f(t)}{h},$$

where $f(t+h)$ is set equal to $f(1)$ for $t+h>1$ and to $f(0)$ if $t+h<0$.

(b) Prove that the set

$$A := \big\{(f,t) \in B\times[0,1] \ : \ f \text{ è differenziabile in } t\big\}$$

is a Borel subset of $B\times[0,1]$.

5. Let X be an infinite dimensional vector space.

(a) Prove that, if X is a Hilbert space, then there exists $r>0$ such that there are infinie disjoint balls of radius r contained in the unit ball of X.

(b) Does the previous claim remain true if we only assume X to be a Banach space.

6. Let $p \in (1,+\infty)$ and consider the Banach spaces $L^p((0,1))$ and $L^p(\mathbb{R})$. Is it true that these are isometrically isomorphic (i.e., that there is a norm-preserving linear bijection between them)?

7. Consider the Cauchy problem

$$\begin{cases} \partial_x u(x,y) + \partial_y u(x,y) = u(x,y)2 & x\in\mathbb{R}, y>0, \\ u(x,0) = v(x) & x\in\mathbb{R}. \end{cases}$$

For which initial conditions $v \in C^\infty(\mathbb{R})$ does there exist a solution $u \in C^1(\mathbb{R}\times[0,+\infty))$?

Hint: consider the evolution of u on the half lines parallel to $x=y$.

8. Show that, if $C \subset L^p((0,1))$, $1<p<\infty$, is closed and convex, then for all $f \in L^p((0,1))$ there exists a unique $\bar{g} \in C$ such that

$$\|f-\bar{g}\|_p = \min_{g\in C} \|f-g\|_p.$$

9. Let $p_1, p_2 \in [1,\infty)$ and assume that $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$|\phi(s)| \le c_1 + c_2 |s|^{p_1/p_2} \quad \text{for every } s \in \mathbb{R}.$$

Prove that, if $f_n \to f$ in $L^{p_1}((0,1))$, then $\phi(f_n) \to \phi(f)$ in $L^{p_2}((0,1))$.

10. Let H be a Hilbert space and let $T\colon H \to H$ be a bounded linear operator such that

$$(Tx, x) \geq \|x\|^2 \quad \text{for every } x \in H,$$

where $(\cdot, \cdot)$ and $\|\cdot\|$ are the scalar product and the norm in H.

(a) Prove that the inverse operator T^{-1} exists and is continuous and bounded.

(b) Let $K\colon H \to H$ be a compact linear operator. Show that, if $T + K$ is one-to-one, then $T + K$ is onto.

Numerical Analysis

11. Let Ω be an open bounded domain in $\mathbb{R}^2$ with regular boundary $\partial\Omega$. Consider the following problem

$$\begin{cases} -\nabla \cdot (\mu \nabla u) + \mathbf{b} \cdot \nabla u + \sigma u = f & \text{in } \Omega \\ u = \overline{u} & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$, $\mathbf{b}$ is a constant vector in $\mathbb{R}^2$, μ is a positive constant, and σ is a real number.

(a) Derive the weak formulation of (1);

(b) Discuss under which assumptions on $\mathbf{b}, \sigma, f$, the bilinear form associated with the weak formulation is coercive;

(c) Write the generic element of the finite elements system matrix A_{ij} and right hand size f_i. Is A symmetric?

12. Let K be a triangle in $\mathbb{R}^2$. Denote by A the area of K and by m_1, m_2 and m_3 the mid-points of its edges. Let $\mathcal{P}_n$ be the set of polynomials of degree n on K. Prove that the following quadrature rule is exact for every polynomial $p \in \mathcal{P}_2$:

$$\int_K p\,(x)\;dx = \frac{1}{3} A\left(p\,(m_1) + p\,(m_2) + p\,(m_3)\right).$$

13. Let $\Omega = (0, 1)$. Consider the following problem

$$\begin{cases} -\mu u'' + bu' = -b & \text{in } \Omega \\ u(0) = u(1) = 0 \end{cases} \tag{2}$$

where the diffision coefficent $\mu > 0$ and the transport coefficient b are constant.

(a) Under which assumpitions on b and μ the solution develops a boundary layer? In proximity of which part of the boundary of the domain would such a boundary layer be located?

(b) Find the analytic solution of such problem.

(c) Write the numerical formulation of the problem obtained using a finite difference scheme. Discuss under which condition(s) the solution of the discretized problem might incur in stability problems, and propose possible numerical remedies.

14. For every 2×2 matrix B let $\|B\|$ indicate the matricial norm induced by the euclidean norm in $\mathbb{R}^2$, i.e.,

$$\|B\| := \sup_{|v|=1} \frac{|Bv|}{|v|}.$$

Given an arbitrary triangle K, define

$$h_K := \max_{x,y\in K} |x-y|$$
$$\rho_K := \max_{B_\rho \subset K} 2\rho,$$

where $B_\rho \subset K$ is a ball of radius ρ contained in K.

Let $\hat{K}$ and K be two triangles in $\mathbb{R}^2$.

(a) Show that there exists an affine and invertible transformation T_K that maps $\hat{K}$ into K;

(b) Calling $B := DT_K$ the Jacobian of the transformation T_K, show that

$$\|B\| \leq \frac{h_K}{\rho_{\hat{K}}}$$
$$\|B^{-1}\| \leq \frac{h_{\hat{K}}}{\rho_K}.$$

15. Let $A \in \mathbb{R}^{m\times n}$, $m \geq n$, be a full rank matrix.

(a) Show that the component x of the solution to the system

$$M\begin{pmatrix} -r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}, \qquad M := \begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix} \in R^{(m+n)\times(m+n)}$$

minimizes $\|Ax-b\|_2$.

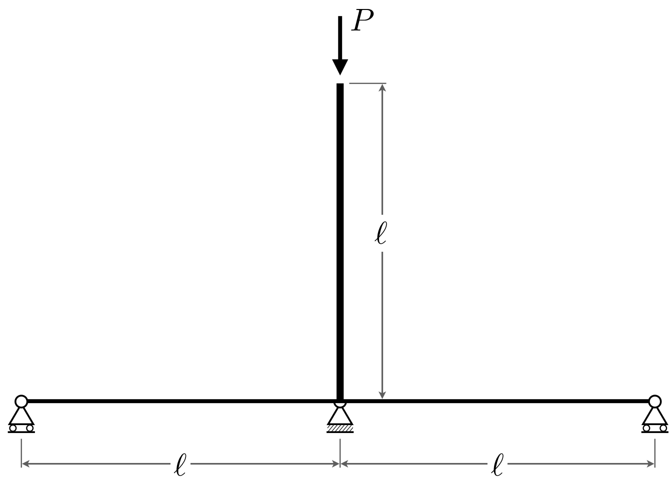
(b) Express the condition number of M in terms of the singular values of A.

(c) Write down an explicit expression for M^{-1} in terms of A and A^T.

Continuum Mechanics

16. Let x be the motion of a continuum body defined in Cartesian components by $x_1 = X_1 \exp(t^2), x_2 = X_2 \exp(t)$, and $x_3 = X_3 + kt$, where X_i, $i = \{1, 2, 3\}$, are the Cartesian coordinates of a material point, t is time and $k > 0$ is a constant. Choose k such that the streamline, at time $t = 1$, that crosses the point (y_1, y_2, y_3) also crosses the point $(2y_1, \exp(\ln(2)/2)y_2, 2y_3)$, with $y_1 > 0, y_2 > 0$, and $y_3 > 0$.

17. Compute the critical load P_c for the planar elastic system shown in the figure. In particular, assume that the vertical bar is rigid and that the horizontal elastic rod is of constant bending stiffness B.



18. Consider a cylindrical body of height H_0 with circular basis of radius R_0. Let $\{\boldsymbol{E}_R, \boldsymbol{E}_\Theta, \boldsymbol{E}_Z\}$ and $\{\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z\}$ be cylindrical polar bases in the reference and in the current configuration, respectively. Compute the deformation gradient and the principal stretches corresponding to a deformation of simple torsion such that $r = R$, $\theta = \Theta + kZ$, and $z = Z$.

19. An elastic strip of length L and rectangular cross-section is subject at its extremities to tensile normal tractions of constant magnitude σ under plane strain conditions. Compute the elongation of the strip using linear elasticity theory and assuming isotropic material response.

20. Let $\mathcal{B}_t$ be the current configuration of a continuum body at time t and let $\mathbf{F}$ and $\mathbf{v}$ be the deformation gradient and the spatial description of the velocity field associated with the motion, respectively. Denoting with $\mathbf{n}(x, t)$ the outward unit normal field to the boundary $\partial\mathcal{B}_t$, show that

$$\dot{\mathbf{n}} = [\mathbf{n} \cdot (\operatorname{grad} \mathbf{v})\mathbf{n}]\, \mathbf{n} - (\operatorname{grad} \mathbf{v})^{\mathrm{T}} \mathbf{n}\,,$$

where the superposed dot stands for the material time derivative. Recall that $\mathbf{n} = \mathbf{F}^{-\mathrm{T}}\mathbf{m}/|\mathbf{F}^{-\mathrm{T}}\mathbf{m}|$, with $\mathbf{m}$ the outward reference unit normal to the boundary of the body.