SISSA – Mathematics Area

Entrance examination for the course in Mathematical Analysis, Modelling, and Applications

September 10, 2020

Solve THREE of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than three) will be considered for the selection.

Mathematical Analysis

Ex. 1. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be 2π periodic functions.

(a) If $f \in C^\infty$, prove that for any $n \in \mathbb{N}$, there exists a constant $C_n > 0$ such that

$$\left|\widehat{f}_k\right| \le \frac{C_n}{|k|^n}, \qquad \forall k \in \mathbb{Z} \setminus \{0\},$$

where $\widehat{f}_k := (2\pi)^{-1} \int_0^{2\pi} f(t)\, e^{-\mathrm{i}kt} dt$ is the k-th Fourier coefficient of f.

(b) For $f \in C^\infty$ and $g \in L^\infty$, prove that

$$\lim_{n\to\infty} \int_0^{2\pi} f(t)\, g(nt)\, \mathrm{d}t = 2\pi \widehat{f}_0\, \widehat{g}_0.$$

(c) Prove that the same result holds true when $f \in L^1$.

Ex. 2. On the two-dimensional periodic domain $\mathbb{T}^2 := (\mathbb{R}/2\pi\mathbb{Z})^2$ consider a C^∞ divergence-free velocity field $u : \mathbb{T}^2 \to \mathbb{R}^2$, namely $u : \mathbb{R}^2 \to \mathbb{R}^2$, $u(x) = (u_1(x_1, x_2), u_2(x_1, x_2))$, is 2π-periodic in each variable x_1, x_2 and $\operatorname{div}(u) = \partial_{x_1} u_1 + \partial_{x_2} u_2 = 0$. Let $f(t, x)$ and $f^\nu(t, x)$ be real valued C^∞ solutions, defined for any time $t \ge 0$, $x \in \mathbb{T}^2$, of the partial differential equations

$$\begin{cases} \partial_t f + u(x) \cdot \nabla f = 0\,, \\ f(0, x) = f^{in}(x)\,, \end{cases} \tag{1}$$

and

$$\begin{cases} \partial_t f^\nu + u(x) \cdot \nabla f^\nu = \nu \Delta f^\nu, \\ f^\nu(0, x) = f^{in}(x)\,, \end{cases} \tag{2}$$

where $\nu > 0$ and the function $f^{in}(x) \in C^\infty(\mathbb{T}^2, \mathbb{R})$ has zero mean, i.e.

$$\int_{\mathbb{T}^2} f^{in}(x) \mathrm{d}x = 0\,.$$

Prove the following:

(a) for any $t \geq 0$, one has

$$\int_{\mathbb{T}^2} f(t,x)\,\mathrm{d}x = \int_{\mathbb{T}^2} f^\nu(t,x)\,\mathrm{d}x = 0\,;$$

(b) for any $t \geq 0$, one has

$$\|f(t)\|_{L^2} = \Big(\int_{\mathbb{T}^2} |f(t,x)|^2 dx\Big)^{\frac{1}{2}} = \left\|f^{in}\right\|_{L^2}\,;$$

(c) there exists a constant $c > 0$ such that, for any $t \geq 0$,

$$\|f^\nu(t)\|_{L^2} \leq e^{-c\nu t}\left\|f^{in}\right\|_{L^2}\,.$$

Ex. 3. Prove that, for any $c \in [0,1)$, the solution $t \mapsto x_c(t)$ of the problem

$$\begin{cases} \ddot{x} + (1+c^2)x - 2c^2x^3 = 0 \\ x(0) = 0 \\ \dot{x}(0) = 1\,, \end{cases}$$

is periodic. What about $c = 1$?

Ex. 4. Let $(f_n)_n$ be a sequence of functions in $L^1[0,1]$ with $f_n \to f$ a.e. and

$$\lim_{n\to\infty} \|f_n\|_{L^1} = \|f\|_{L^1}\,. \tag{3}$$

(a) Prove that $\lim_{n\to\infty} \|f_n - f\|_{L^1} = 0$.

(b) Prove with an example that, if the condition (3) is changed in

$$\|f_n\|_{L^1} \quad \text{converges}$$

the thesis is not true.

Ex. 5. Let ℓ^2 and ℓ^∞ denote the sequence spaces $\ell^2 := \big\{x = (x_i)_{i\in\mathbb{N}}\,,\, x_i \in \mathbb{R} : \|x\|_2 := (\sum_{i=1}^\infty |x_i|^2)^{1/2} < \infty\big\}$ and $\ell^\infty := \big\{x = (x_i)_{i\in\mathbb{N}}\,,\, x_i \in \mathbb{R} : \|x\|_\infty := \sup_{i=1,\ldots,\infty} |x_i| < \infty\big\}$.

(a) Given any $w = (w_i)_{i\in\mathbb{N}} \in \ell^\infty$, determine the spectrum $\sigma(T_w)$ of the multiplicative operator $T_w : \ell^2 \to \ell^2$ associating to any sequence $x = (x_i)_{i\in\mathbb{N}} \in \ell^2$ the element Tx with components $(Tx)_i = x_i w_i$, $\forall i \in \mathbb{N}$;

(b) Let $A \subset \mathbb{C}$ be an arbitrary non-empty compact set. Construct a linear and bounded operator $T : \ell^2 \to \ell^2$ such that its spectrum $\sigma(T)$ coincides with A, i.e. $\sigma(T) = A$.

Ex. 6. Consider a differential equation

$$\dot{x} = f(t,x)\,, \quad x \in \mathbb{R}^n\,, \tag{4}$$

where the vector field $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz continuous and there exist constants $C > 0$ and $M > 1$ such that

$$|f(t,x)| \leq C|x|^M\,, \quad \forall x \in \mathbb{R}^n\,,\ t \in \mathbb{R}\,.$$

Consider the solution $x_\varepsilon(t)$ of (4) with initial datum $x(0) = \varepsilon$ and let $(-T_{\max}(\varepsilon), T_{\max}(\varepsilon))$ be its maximal interval of definition.

Prove that there exists $\varepsilon_0 > 0$ and a constant $c > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, it results

$$T_{\max}(\varepsilon) \geq c\varepsilon^{-(M-1)}\,.$$

Numerical Analysis

Ex. 7.

(a) Consider the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ with

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Consider then the iterative method $\boldsymbol{x}^{(k+1)} = \boldsymbol{B}(\theta)\boldsymbol{x}^{(k)} + \boldsymbol{g}(\theta)$, in which

$$\boldsymbol{B}(\theta) = \frac{1}{4}\begin{bmatrix} 2\theta^2 + 2\theta + 1 & -2\theta^2 + 2\theta + 1 \\ -2\theta^2 + 2\theta + 1 & 2\theta^2 + 2\theta + 1 \end{bmatrix}, \qquad \boldsymbol{g}(\theta) = \begin{bmatrix} \frac{1}{2} - \theta \\ \frac{1}{2} - \theta \end{bmatrix}.$$

Prove that the iterative method is consistent $\forall \theta \in \mathbb{R}$, and discuss its convergence for a fixed value of θ.

(b) Suppose instead that the system matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is a large real sparse matrix, $n \gg 1$. Describe an iterative method to numerically solve the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ for some $\boldsymbol{b} \in \mathbb{R}^n$. You may add further condition on $\boldsymbol{A}$ if required by the numerical method.

(c) Illustrate a few available stopping criteria for the numerical method described in (b), and highlight the main properties of each criterion.

Ex. 8.

(a) Let $I \subset \mathbb{R}$ be a closed interval. Let $f : I \to \mathbb{R}$ be a continuous function. Assume that there exists at least a value $\alpha \in I$ such that $f(\alpha) = 0$. Discuss a numerical method to approximate α. You may add further regularity assumptions, if required by the numerical method.

(b) Let $\Gamma \subset \mathbb{R}^n$ be a compact set. Let $\mathbf{g} : \Gamma \to \mathbb{R}^n$ be a continuous function. Assume that there exists at least a value $\mathbf{v} \in \Gamma$ such that $\mathbf{g}(\mathbf{v}) = \mathbf{0}$. Discuss a numerical method to approximate $\mathbf{v}$, and highlight similarities and/or differences with the answer provided in (a). You may add further regularity assumptions, if required by the numerical method.

(c) Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Let $u : \Omega \to \mathbb{R}$ be the solution to $n(u) = 0$, where $n(u) = -\Delta u + u^3 - 1$, and u is such that $u = 0$ on $\partial\Omega$. Discuss a numerical method to approximate u, and highlight how you make use of the answer at (b).

Ex. 9. Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Define the following problem

$$\begin{cases} -\mathrm{div}\,(\mu \nabla u) + \mathbf{b} \cdot \nabla u = f, & \text{in } \Omega \\ u = 0, \text{ on } \partial\Omega, \end{cases}$$

where $\mu \in \mathbb{R}$, $\mathbf{b} \in \mathbb{R}^2$ and $f : \Omega \to \mathbb{R}$.

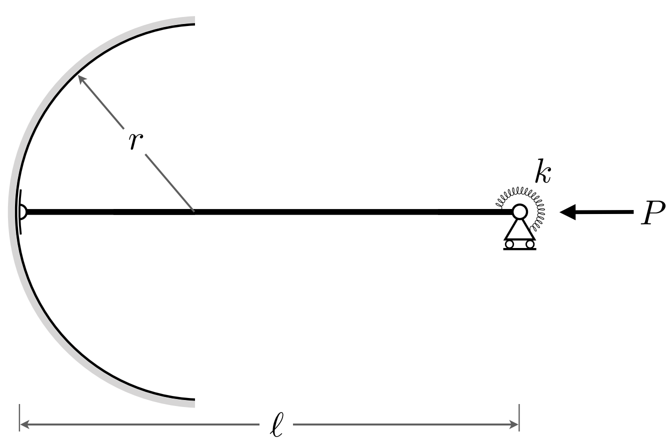
(a) Discuss under which assumptions on the data $\mu, \mathbf{b}$ and f the problem is well posed, and derive the weak formulation.

(b) Let $|\mathbf{b}| \ll \mu$: introduce a finite element approximation u_h to u, and discuss convergence properties of u_h to u as the finite element mesh size $h \to 0$.

(c) Let $|\mathbf{b}| \gg \mu$: introduce a finite element approximation u_h to u. Describe which (if any) numerical difficulties arise in this case, especially if $h \gg 0$. Do you expect the convergence properties that you mentioned in (b) to hold in this case as well?

Continuum Mechanics

Ex. 10. Compute the critical load, P_c, of the elastic system shown in the figure. In particular, this comprises a rigid bar of length ℓ that is constrained on its left extremity by a smooth circular profile of radius r. In the figure, k denotes the stiffness of the torsional spring.



Ex. 11. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis. Consider a cylindrical solid, of axis $\mathbf{e}_3$, at equilibrium in a state of uniaxial tension under prescribed normal tractions of magnitude σ acting on its bases. Assume that the components $E_{\alpha\alpha} = \mathbf{e}_\alpha \cdot \mathbf{E}\,\mathbf{e}_\alpha$ and $E_{\beta\beta} = \mathbf{e}_\beta \cdot \mathbf{E}\,\mathbf{e}_\beta$ of the (infinitesimal) strain are given, where $\mathbf{e}_\alpha = \cos\alpha\,\mathbf{e}_2 + \sin\alpha\,\mathbf{e}_3$ and $\mathbf{e}_\beta = \cos\beta\,\mathbf{e}_2 + \sin\beta\,\mathbf{e}_3$, with $\alpha \neq \beta$. Determine the Young's modulus and the Poisson's ratio using linear elasticity and assuming isotropic material response.

Ex. 12. A cylindrical chalk stick with circular cross section of radius r is subject at its extremities to applied bending moments of magnitude M. Let σ_f be the strength of chalk (the force per unit area defining the strength of the material). Estimate the magnitude M_f of the bending moments at which the chalk stick breaks. Recall that the moment of inertia of a circular cross section is $\pi r^4/4$.