SISSA – Mathematics Area

Entrance examination for the course in Mathematical Analysis, Modelling, and Applications

June 9, 2020

Solve THREE of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than three) will be considered for the selection.

Mathematical Analysis

1. How many solutions of the differential equation $x'(t) = x(t) - e^{-t^2}$ satisfy the conditions $\lim\limits_{t\to-\infty} x(t) = \lim\limits_{t\to+\infty} x(t) = 0$?

2. Find all solutions of the Cauchy problem

$$(x'(t))^3 = 8tx(t), \quad x(0) = 0\,,$$

such that $x(t) \neq 0$ for any $t \neq 0$.

3. Let f_n be a sequence of $L^2([0,1])$ such that

$$f_n \rightharpoonup f \text{ weakly in } L^2([0,1]) \text{ and } \|f_n\|_{L^2} \to \|f\|_{L^2}.$$

Prove that $f_n \to f$ strongly in $L^2([0,1])$.

4. Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{n^n}$ converges or diverges.

5. Let f_n be a sequence in $L^2([0,1])$ such that $f_n \rightharpoonup f$ weakly in $L^2([0,1])$ and let

$$F_n(t) := \int_0^t f_n(s)ds \quad \text{and} \quad F(t) := \int_0^t f(s)ds.$$

Prove that

(a) $F_n(t) \to F(t)$ for every $t \in [0,1]$;

(b) there exists a constant $c \in \mathbb{R}$ such that $|F_n(t_2) - F_n(t_1)| \leq c|t_2 - t_1|^{1/2}$ for every t_1, $t_2 \in [0,1]$;

(c) $F_n \to F$ uniformly in $[0,1]$.

6. Let X be a Hilbert space and let A_n, $n \in \mathbb{N}$, and A be continuous linear operators from X into X. Assume that for every x, $y \in X$ we have $(A_n x, y) \to (Ax, y)$ as $n \to \infty$, where $(\cdot,\cdot)$ denotes the scalar product in X. Prove that

(a) there exists a constant $c \in \mathbb{R}$ such that $\|A_n\| \le c$ for every $n \in \mathbb{N}$, where $\|\cdot\|$ is the operator norm;

(b) $(A_n x_n, y_n) \to (Ax, y)$ for every x, $y \in X$ and for every pair of sequences x_n and y_n strongly converging to x and y, respectively.

Numerical Analysis

7. Consider the following non-singular matrix $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 5 & 6 & 0 \end{bmatrix}$$

(a) Compute the LU factorization of the matrix $\boldsymbol{A}$;

(b) Exploit the LU factorization of $\boldsymbol{A}$, and compute its inverse $\boldsymbol{A}^{-1}$.

8. One wants to solve the equation $x + \ln x = 0$, whose root is $r \sim 0.5$, using one or more of the following iterative methods: $\{i\}$ $x_{k+1} = -\ln x_k$ $\{ii\}$ $x_{k+1} = e^{-x_k}$ $\{iii\}$ $x_{k+1} = \frac{x_k + e^{-x_k}}{2}$

(a) Which of the three methods can be used?

(b) Which method should be used?

(c) Give an even better iterative formula.

9. Determine the exact solution to the following Eigenvalue problem:

$$y'' + \lambda y = 0 \qquad \text{with} \qquad y(0) = y(1) = 0.$$

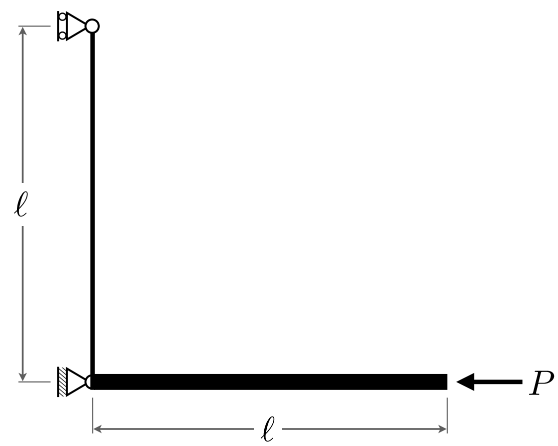
Let $h = 1/N$. The problem approximated by finite differences is written as:

$$y_{n+1} - 2y_n + y_{n-1} + h^2 \lambda_h y_n = 0, \qquad n = 1, 2, \ldots, N-1$$
$$y_0 = y_N = 0.$$

Knowing that the components of the eigenvectors are the values of the eigenfunctions in nodes $x = nh$, study the convergence of the eigenvalues λ_h and estimate the error.

Continuum Mechanics

10. Compute the critical load P_c for the elastic system shown in the figure. In particular, assume that the horizontal bar is rigid and that the vertical elastic rod is of constant bending stiffness B.



11. Consider a right prismatic solid of height h and square bases of edge b, at equilibrium under prescribed normal tractions on the lateral surface. In particular, let $\sigma \boldsymbol{n}$ and $-2\sigma \boldsymbol{n}$ be the tractions acting on mutually parallel faces ($\boldsymbol{n}$ denotes the outward unit normal to the lateral faces). Compute the Cauchy stress tensor. Using linear elasticity and assuming isotropic material response, determine σ corresponding to an elongation of the solid of $h/10$.

12. Compute the mechanical response (relation between applied force F, elongation δ, and their time derivatives) of the one-dimensional rheological model shown in the figure. Notice that the system comprises two linear elastic springs and a linear dashpot.