SISSA – Mathematics Area

Entrance examination for the course in Mathematical Analysis, Modelling, and Applications

February 14, 2023

Solve FIVE of the following problems. In the first page of your examination paper please *write neatly* the list of the exercises you have chosen. These exercises only (in any case not more than five) will be considered for the selection.

Mathematical Analysis

1. Let A, B be bounded operators on a Hilbert space. Prove or disprove the following sentences:

(i) If AB is compact, at least one between A and B is compact.

(ii) If AB is injective, at least one of the two is injective.

(iii) If AB is surjective, at least one of the two is surjective.

(iv) If AB is bijective, at least one of the two is bijective.

2. Consider the operator $T\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined, for a sequence $x \equiv (x_k)_{k\in\mathbb{Z}}$, by

$$(Tx)_k := x_{k+1} - 2x_k + x_{k-1}.$$

Discuss, properly justifying the answers, the following questions:

(i) Is T bounded? Which is its norm?

(ii) Is T injective? Is it surjective? Is it invertible?

(iii) Is T compact?

(iv) Which is the spectrum of T?

3. Show that

$$\lim_{n\to+\infty} \int_0^{2\pi} |\sin(nx)f(x)|dx = \frac{2}{\pi}\int_0^{2\pi} |f(x)|dx \text{ for any } f \in L^1(0, 2\pi).$$

(Hint: consider first the case where f is the characteristic function of an interval.)

4. Answer the following questions.

(i) Consider the embedding $L^p(0, 2\pi) \hookrightarrow L^q(0, 2\pi)$ for $p \geq q$. Is there any choice of p and q such that the embedding is a compact operator?

(ii) Let $I = [a, b]$ be a bounded interval: is the embedding of $C^1([a, b]) \hookrightarrow C([a, b])$ compact?

(iii) Let $C_b(\mathbb{R})$ be the space of continuous bounded functions f over $\mathbb{R}$ with norm $\|f\|_{C_b} = \sup_{x\in\mathbb{R}} |f(x)|$, and let $C_b^1(\mathbb{R})$ be the space of continuous bounded functions f over $\mathbb{R}$ with continuous bounded derivative f' with norm $\|f\|_{C_b} + \|f'\|_{C_b}$. Is the embedding $C_b^1(\mathbb{R}) \hookrightarrow C_b(\mathbb{R})$ compact?

The candidate must justify the answers.

5. Answer the following questions.

(i) Let $f \in C^0(\mathbb{R}) \cap L^1(\mathbb{R})$. Prove that, if the limit

$$\lim_{x\to+\infty} f(x)$$

exists, then it must be equal to 0.

(ii) Find an example of a function $f \in C^0(\mathbb{R}) \cap L^1(\mathbb{R})$ such that the limit

$$\lim_{x\to+\infty} f(x)$$

does not exist.

(iii) Let $f \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$ with $f' \in C^0(\mathbb{R}) \cap L^1(\mathbb{R})$. Prove that

$$\lim_{x\to+\infty} f(x) = 0\,.$$

6. Let $f\colon [0, +\infty) \to [0, +\infty)$ be a concave function with $f(0) = 0$.

(i) Prove that

$$f(x + y) \leq f(x) + f(y)$$

for every $x, y \in [0, +\infty)$.

(ii) Prove that

$$\frac{f(x)}{x} \geq \frac{f(y)}{y}$$

for every $x, y \in [0, +\infty)$ with $x \leq y$.

(iii) Prove that the limit

$$\lim_{x\to+\infty} \frac{f(x)}{x}$$

exists and is finite.

(iv) Assume, in addition, that $f \in C^1((0,+\infty))$. Prove that

$$\lim_{x\to+\infty} f'(x) = \lim_{x\to+\infty} \frac{f(x)}{x}.$$

7. Let $I := [0,1]$ and let $I_k^i := [\frac{i-1}{k}, \frac{i}{k}]$ for $k \in \mathbb{N}$ and $i = 1, \ldots, k$. For every $A \subset I$ let $\chi_A \colon I \to \mathbb{R}$ be the characteristic function of A, defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \in I \setminus A$. For every $k \in \mathbb{N}$ let $T_k \colon L^1(I) \to L^1(I)$ be the linear operator defined by

$$(T_k(f))(x) := k \sum_{i=1}^{k} \chi_{I_k^i}(x) \int_{I_k^i} f(y)dy$$

for every $f \in L^1(I)$.

(i) Prove that

$$\|T_k(f)\|_{L^1(I)} \le \|f\|_{L^1(I)}$$

for every $f \in L^1(I)$.

(ii) Prove that

$$T_k(f) \to f \quad \text{in } L^1(I)$$

for every $f \in C^0(I)$.

(iii) Prove that

$$T_k(f) \to f \quad \text{in } L^1(I)$$

for every $f \in L^1(I)$.

(iv) Is it true that

$$\lim_{k\to+\infty} \sup_{\substack{f\in L^1(I) \\ \|f\|_{L^1(I)}\le 1}} \|T_k(f) - f\|_{L^1(I)} = 0\,?$$

8. Let $f \in L^1(\mathbb{R})$ with $f \ge 0$ a.e. in $\mathbb{R}$ and let $g \colon \mathbb{R} \to \mathbb{R}$ be the function defined by

$$g(x) := \int_{x^2+\sin(x)}^{x^4+2} f(t)\,dt$$

for every $x \in \mathbb{R}$. Prove that g has a maximum point in $\mathbb{R}$.

9. Let $\alpha \in [0,\infty)$ and consider the differential equation

$$\ddot{x}(t) + x(t) - \sin(\alpha x(t)) = 0\ .$$

(i) Prove that if $\alpha \in [0, 1]$ then all the solutions are periodic.

(ii) Prove that if $\alpha > 1$ there are solutions not periodic.

10. Find a C^1 solution $u : \mathbb{R}^2 \to \mathbb{R}$ to the PDE

$$u_x + u_y = u, \quad u(0, x) = \sin(x).$$

Numerical Analysis

11. Consider the nonlinear equation $5x^4 - 4x^2 + 2x - 3 = 0$. We are interested in the root $x_0 \approx -1.25$.

(i) Given a user defined tolerance ε and established a stopping criterion, use the bisection method and the Newton method to obtain a numerical approximation of x_0.

(ii) Which method requires a lower number of iterations?

(iii) Could you derive a theoretical prediction of the number of iterations to be performed to converge? Is such prediction applicable to both methods and to any stopping criterion?

12. Consider the following non-singular matrix $\mathbf{M} \in \mathbb{R}^{3\times3}$:

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 6 \\ 5 & 7 & 0 \end{bmatrix}$$

(i) Compute the LU factorization of the matrix $\mathbf{M}$.

(ii) Exploit the LU factorization of $\mathbf{M}$ and compute its inverse $\mathbf{M}^{-1}$.

(iii) Compute the p-norm $||\mathbf{M}||_p$ with $p = 1$ and $p = \infty$.

13. Consider the linear system $\mathbf{Ax} = \mathbf{b}$.

(i) Write a pseudo code that employs the Gauss-Seidel method and the Jacobi method to solve the mentioned system.

(ii) Discuss the convergence of two methods in the following two cases:

- $\mathbf{A}$ is strictly diagonally dominant by row;
- $\mathbf{A}$ is symmetric positive definite.

14. Consider the following PDE governing an unsteady one-dimensional heat conduction problem:

$$\frac{\partial \varphi}{\partial t} - k\frac{\partial^2 \varphi}{\partial x^2} = 0 \quad \text{for} \quad (x,t) \in (0,L) \times (t_0,T),$$

where $\varphi(x,t)$ is the temperature and $k \in \mathbb{R}^+$ is the thermal conductivity coefficient.

(i) Based on the assumption that $\varphi(x,t) = a(x)b(t)$ derive the ODE whose unknown is $b(t)$ (*Suggestion*: if you have $\alpha(t) = \beta(x)$ where $\alpha(t)$ and $\beta(x)$ are two time-dependent and space-dependent only functions respectively, you can impose $\alpha(t) = \beta(x) = K$ with $K \in \mathbb{R}$).

(ii) Consider the ODE obtained at point (i), set $b(t_0) = b_0$ and use the forward Euler method with stepsize Δt to discretize it.

(iii) Finally discuss the stability of the scheme.

15. Consider the linear advection equation:

$$\frac{\partial \varphi}{\partial t} + a\frac{\partial \varphi}{\partial x} = 0 \quad \text{for} \quad (x,t) \in (0,L] \times (t_0,T),$$

where $a \in \mathbb{R}^+$. The equation is endowed with the boundary condition $\varphi(0,t) = \varphi_0$ and initial condition $\varphi(x,t_0) = \varphi^0(x)$. Let Δt and Δx be the time step and the mesh size respectively.

(i) Within a finite difference method, derive

- a first order forward approximation of $\dfrac{\partial \varphi}{\partial t}$,
- a first order backward approximation of $\dfrac{\partial \varphi}{\partial x}$,

and use them to discretize the differential problem.

(ii) Discuss the stability of the scheme.

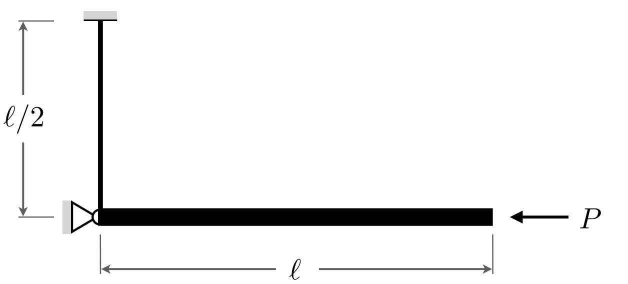
(iii) Compute the amount of artificial viscosity introduced by the scheme.

Continuum Mechanics

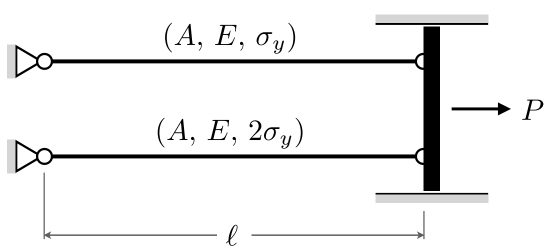
16. Consider the homogeneous deformation $y : \mathcal{B} \to \mathcal{E}$ such that (in Cartesian components) $y_1 = \alpha\, x_1$, $y_2 = \beta\, x_2$, and $y_3 = x_3$, where α and β are positive scalars and x_i, with $i = \{1, 2, 3\}$, are the coordinates of a material point $\mathbf{x} \in \mathcal{B}$. Compute the right stretch tensor and the deformed length of the material curve $c(\sigma) : [0, 1] \to \mathcal{E}$ with $x_1 = \sigma$, $x_2 = \sigma^2$, and $x_3 = 0$.

17. An elastic strip of length ℓ and rectangular cross section of area A is subject at its extremities to tensile normal tractions of constant magnitude σ. The strip is constrained such that transverse displacements are prohibited on two opposite lateral faces. Assuming linear elastic, isotropic and incompressible material response, determine the stress and strain tensors and compute the elongation of the strip corresponding to the total force σA.

18. Compute the critical load P_{c} for the elastic system shown in the figure. Assume that the horizontal bar is rigid and that the vertical elastic rod is of constant bending stiffness B.



19. Consider the mechanical system shown in the figure, consisting of two elastic-perfectly plastic rods of cross-sectional area A, Young's modulus E, and yield stresses σ_y and $2\sigma_y$, respectively. Compute the mechanical response of the system under the prescribed external force P (*i.e.*, the external force versus elongation behaviour) and the magnitude of the external force corresponding to plastic collapse.



20. A cylindrical chalk stick with circular cross section of radius r is subject at its extremities to applied torques and axial forces of magnitude T and N, respectively. Let σ_{f} be the strength of chalk. Determine the loading conditions at which the chalk stick breaks.