SISSA – Mathematics Area

Entrance examination for the course in Mathematical Analysis, Modelling, and Applications

March 5, 2024

Solve FIVE of the following problems. Mark in the table below the exercises you have chosen. These exercises only will be considered for the selection.

Mark here the FIVE problems you have chosen																			
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Mathematical Analysis

1. (*i*) Show that the map

$$T : C^0([0,1]) \to C^0([0,1]), \qquad Tf(x) := f(x) + \int_0^x f(y)dy,$$

defines an isomorphism of vector spaces of $C^0([0,1])$ with itself.

(*ii*) Prove that there exists $\varepsilon_0 > 0$ such that, for any $g \in C^0([0,1])$ with $\|g\|_{C^0} \leq \varepsilon_0$, there is at least one solution $f \in C^0([0,1])$ to the nonlinear equation

$$Tf + f^2 = g\ .$$

2. Let $y(t)$ be a solution of the differential equation

$$y'' = y^2 + (y')^2$$

not identically 0. Prove that $y(t)$ cannot be globally defined on $\mathbb{R}$.

3. Consider the operator $T : L^2([0,1]) \to L^2([0,1])$ given by

$$Tf(x) = \int_0^1 xy\,(1 - xy)\,f(y)\,dy$$

(a) Show that T is continuous.

(b) Determine the spectrum of T.

4. Given a Banach space B, a sequence $n \mapsto v_n \in B$ is called *weakly Cauchy* provided for any $L \in B^*$ the sequence $n \mapsto L(v_n) \in \mathbb{R}$ is Cauchy. B is said *weakly sequentially complete* provided any weakly Cauchy sequence has a weak limit. Determine, with proof, if the following Banach spaces are weakly sequentially complete:

1. ℓ^2
2. $C([0,1])$
3. ℓ^1 (*Hint:* it is)

5. Given $A, B \subset \mathbb{R}^d$ we denote by $A + B \subset \mathbb{R}^d$ the set $\{a + b : a \in A, b \in B\}$.

1. Let $K_1, K_2 \subset \mathbb{R}$ be compact. Prove that $\mathcal{L}^1(K_1 + K_2) \geq \mathcal{L}^1(K_1) + \mathcal{L}^1(K_2)$.

2. Prove that there are compact negligible sets K_1, K_2 such that $K_1 + K_2 \supset [0,1]$.

3. Let $f : [0,1] \to \mathbb{R}$ be a twice continuously differentiable function. Let

$$\mathcal{A} = \{(x,y) \in \mathbb{R}^2 \mid y = f(x), 0 \le x \le 1\}.$$

Prove that $\mathcal{L}^2(\mathcal{A} + \mathcal{A}) = 0$ if and only if f is affine.

Here and below $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure.

6. Let $p, q \in (-\infty, 1) \setminus \{0\}$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g : \mathbb{R} \to (0, \infty)$ be Borel.
Prove that

$$\int fg \, d\mathcal{L}^1 \ge \Big(\int f^p \, d\mathcal{L}^1\Big)^{\frac{1}{p}} \Big(\int g^q \, d\mathcal{L}^1\Big)^{\frac{1}{q}} ,$$

where at the right hand side it is intended that $0 \cdot \infty = 0$.

7. Let $P([0,1])$ be the space of Borel probability measures on $[0,1]$.

i) Let $\mu \in P([0,1])$. Prove that $\mu \ll \mathcal{L}^1$ if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that $f \in C([0,1])$, $0 \le f \le 1$ and $\int f d\mathcal{L}^1 < \delta$ implies $\int f d\mu < \varepsilon$.

ii) Let $\delta > 0$ and consider the functional $F_\delta : P([0,1]) \to [0, \infty]$ given by

$$F_\delta(\mu) := \sup\Big\{ \int f \, d\mu \ : \ f \in C^0([0,1]), \ \ 0 \le f \le 1, \ \ \int_0^1 f \, d\mathcal{L}^1 < \delta \Big\}.$$

Prove that F_δ is lower semicontinuous with respect to the weak* topology on $P([0,1])$ (recall that $P([0,1])$ is canonically isomorphic to the dual of $C^0([0,1])$).

iii) Prove that the collection of measures in $P([0,1])$ absolutely continuous with respect to $\mathcal{L}^1$ is weakly* Borel, i.e. it belongs to the σ-algebra generated by the weak* topology.

8. (i) Prove that there exists a constant $C > 0$ such that for all functions $u \in C^3(0,1)$ and continuous in $[0,1]$ such that $u(0) = u(1) = 0$ and u changes sign in $(0,1)$ we have

$$\int_0^1 |u(x)|^2 dx \le C \int_0^1 |u'''(x)|^2 dx \ .$$

(ii) Prove that there exists a constant $C > 0$ such that for all functions $u \in C^k(0,1)$, $k \in \mathbb{N}$, such that the equation $u(x) = 0$ has at least k solutions we have

$$\int_0^1 |u(x)|^2 dx \le C \int_0^1 |u^{(k)}(x)|^2 dx.$$

9. Let $V\colon \mathbb{R} \to \{-1,0\}$ be defined by $V(0) = -1$ and $V(z) = 0$ if $z \neq 0$. Given $x, v \in \mathbb{R}$, compute

$$\lim_{\rho\to 0^+} \frac{1}{\rho} \min\Big\{ \int_0^\rho (|u'(t)|^2 + V(u(t)))dt : u \in H^1(0,\rho),\ u(0) = x, u(\rho) = x + \rho v \Big\}.$$

(*Hint*: consider the case $x = 0$ separately.)

10. Consider the following system of linear partial differential equations:

$$\begin{cases} \partial_t u + \partial_x u + v = 0, \\ \partial_t v + \lambda \partial_x v = 0, \quad x \in \mathbb{R},\ t > 0, \end{cases}$$

with initial data

$$\big(u(0,x), v(0,x)\big) = \big(u_0(x), v_0(x)\big) \in C^1(\mathbb{R},\mathbb{R}^2) \cap L^1(\mathbb{R},\mathbb{R}^2) \cap L^\infty(\mathbb{R},\mathbb{R}^2).$$

i) Write explicitly the solution $(u(t,x), v(t,x))$ in terms of the initial data (u_0, v_0) and the parameter $\lambda \in \mathbb{R}$.

ii) Find for which values of $\lambda \in \mathbb{R}$ we have

$$\sup_{t\geq 0,\ x\in\mathbb{R}} |u(t,x)| + |v(t,x)| < \infty$$

for any initial data as above such that

$$\|u_0\|_{L^1} + \|v_0\|_{L^1} + \|u_0\|_{L^\infty} + \|v_0\|_{L^\infty} \leq 1.$$

Numerical Analysis

11. A rather classical example of second order ODE is the equation of motion of an ideal pendulum:

$$\theta''(t) = -\sin(\theta) \quad \text{with} \quad \theta(0) = 0 \quad \theta'(0) = 1$$

(a) Write the second order ODE as a system of first-order ODEs.

(b) Write the explicit and implicit Euler schemes for such system. For each scheme, propose an algorithm for advancing the solution from an initial condition at $t = 0$.

(c) Now, let us simplify the formulation and consider the linearised case close to $\theta = 0$ which reads:

$$\theta''(t) = -\theta \quad \text{with} \quad \theta(0) = 0 \quad \theta'(0) = 1$$

Write the explicit Euler and implicit Euler schemes. The discrete version of the system of ODEs can be recast in the form:

$$X^{n+1} = \mathbb{M}X^n.$$

Study the behaviour of:

$$\lim_{n\to+\infty} X^n$$

(d) What happens if we consider the Crank-Nicolson scheme? Which conclusions can we take in terms of stability analysis?

12. Consider the classical transport equation in 1D with periodic boundary conditions:

$$\begin{aligned} &\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \qquad x \in [0;1] \quad t > 0. \\ &u(0,t) = u(1,t) \qquad \forall t > 0 \\ &u(x,0) = f(x) \qquad x \in (0,1), \end{aligned} \tag{1}$$

for given $c \in \mathbb{R} \setminus \{0\}$ and initial datum f.

(a) Give a complete description of the discretisation of problem (1) using the combination of the Crank-Nicolson scheme as time-integrator and second-order centered finite differences for the space derivative.

(b) Write explicitly the matrices A and B such that $AU^{n+1} = BU^n$ for the case of the space grid given by the points $x_j = j/N$ with $N = 3$ and $j = 0, ..., N$ (i.e. discretising the computational domain with 4 collocation points).

(c) Using the Von-Neumann/Fourier spectral analysis or otherwise show that:

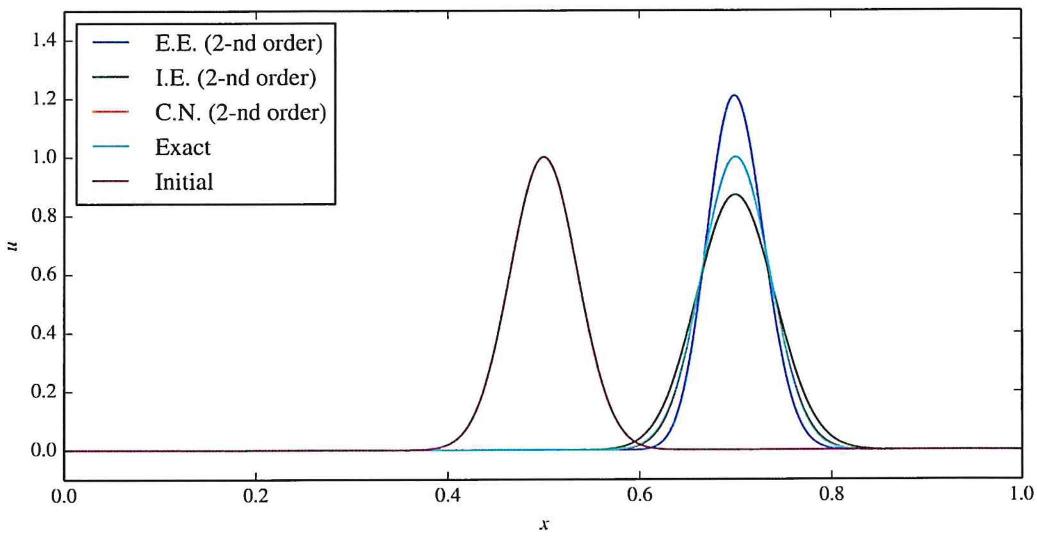
$$|\widehat{u}_k^{n+1}| = |\widehat{u}_k^n|$$

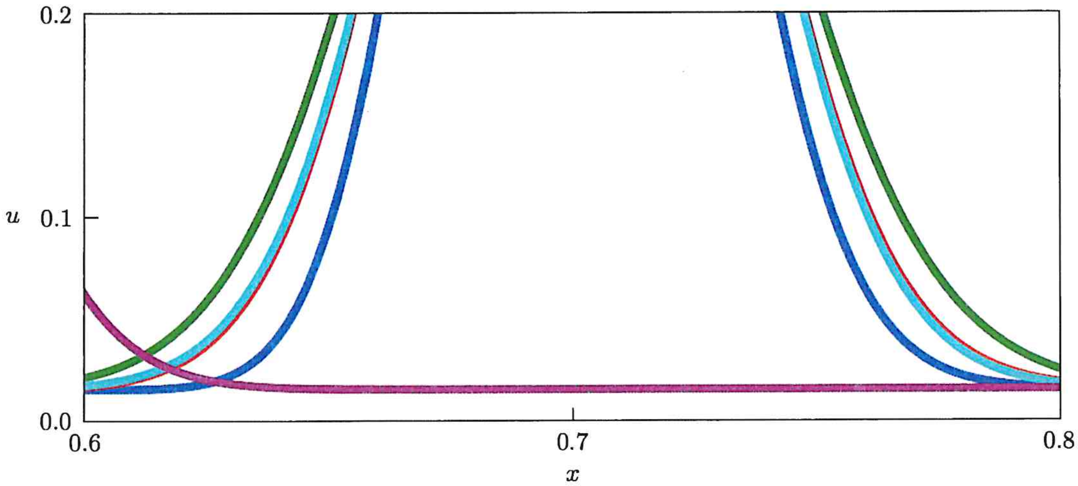
Hint: the solution at the point x_j can be written in Fourier expansion as:

$$u_j^n = \sum_{k=-P}^{P} \widehat{u}_k^n \exp(2\pi i j k \delta x)$$

(d) Problem (1) with $c \equiv 1$ has been solved with initial datum f the gaussian shown in the figure below using the Crank-Nicolson (C.N.) scheme above as well as with the Explicit Euler (E.E.) and Implicit Euler (I.E.) schemes (again coupled with the second-order centered finite difference scheme). The results obtained by the three methods at $T = 0.2$ are shown in the figure below. In particular, the Crank-Nicolson and the exact solution, which are indistinguishable in the figure, can be distinguished in the zoomed figure below. Exhaustively comment the results obtained with each method.

(e) Can you come up with a nodally exact discretisation for problem (1)?





13. A very popular example of ill-conditioned matrix is the so-called Vandermonde matrix. Given a set of scattered data $(\mathbf{x}, \mathbf{y})$, the interpolation problem can be written as: find the coefficients c of a given basis f_i with $i = 1, \ldots, n$, such that the interpolation conditions are satisfied:

$$y_i = \sum_{j=1}^{n} f_j(x_i) c_j$$

This can be written as a linear system $y = Ac$ with $A_{ij} = f_j(x_i)$. If you take the basis of monomials you get the Vandermonde matrix:

$$A_{ij} = x_i^j \quad i,j = 1,...,n.$$

Consider in particular the points $x_1 = 1,\ x_2 = 2,\ x_3 = 3,\ x_4 = 4$.

(a) Write explicitly the matrices A for the cases $n = 2, 3, 4$.

(b) Compute the condition number for each matrix (you can use any norm)

(c) In the case $n = 4$, consider the data:

$$\mathbf{y} = (1, 8, 27, 64)^T$$

and solve the linear system $A\mathbf{c} = \mathbf{y}$

(d) Consider the modified data:

$$\mathbf{y} = (1, 9, 26, 65)^T$$

and solve again the linear system.

(e) Evaluate $||\delta\ \mathbf{y}||_1/||\mathbf{y}||_1$ and $||\delta\mathbf{c}||_1/||\mathbf{c}||_1$ and comment.

14. For $n \in \mathbb{N}$ and $x \in [-1, 1]$, we name $T_n(x) := \cos(n \cos^{-1} x)$ the Chebyshev polynomial of degree n.

(a) Deduce the recurrence relation

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0, \qquad n \geq 1,$$

and, hence, show that for all $n \in \mathbb{N}$ the function T_n is indeed a polynomial of degree n. Prove that the leading term of T_n is given by $2^{n-1}x^n$.

(b) Show that the roots of T_{n+1} are the following points in the interval $[-1, 1]$:

$$x_i = \cos\left(\frac{(2i-1)\pi}{2n}\right) \quad i = 0, \ldots, n. \tag{2}$$

(c) Let $f : [-1, 1] \to \mathbb{R}$ be a function such that $f^{(n+1)}$ is continuous in $[-1, 1]$. Show that there exists a unique polynomial p_n of degree n such that $p_n(x_i) = f(x_i)$, $i = 0, \ldots, n$.

(d) Prove the interpolation error bound

$$\max_{x\in[-1,1]} |f(x) - p_n(x)| \leq \frac{M_{n+1}}{2^n(n+1)!},$$

for some constant M_{n+1} which you should specify.

Hint: You may use without proof the error identity $f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x)$, *where* $\pi_{n+1}(x) = (x - x_0)\dots(x - x_n)$.

(e) Discuss the distribution within $[-1, 1]$ of the Chebyshev interpolation points (2) and its relevance for the polynomial interpolation problem analysed in points (c) and (d).

15. For $\Omega \subset \mathbb{R}^d$ an open and bounded domain with Lipschitz boundary $\partial\Omega$, consider the boundary value problem

$$\begin{aligned} -a\Delta u + cu &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{3}$$

where Δ is the Laplace operator, $f : \Omega \to \mathbb{R}$ is a given function, and $a, c \in \mathbb{R}$ with $a > 0, c \geq 0$.

(a) Assuming $c > 0$, formulate the weak formulation associated to (3): find $u \in V$ such that

$$\mathcal{A}(u, v) = \mathcal{F}(v) \quad \forall v \in V, \tag{4}$$

by specifying $\mathcal{A}$, $\mathcal{F}$ and the (real) Hilbert space V. Clearly indicate for which class of datum f the problem is well-defined. Prove that, when it is well defined, the weak formulation is well-posed, carefully characterising the dependence of the solution on the datum (a priori bound).

(b) Give the relevant details from point (a) above in the case $c = 0$.

(c) Prove that u solves (4) if and only if it minimises in V the quadratic functional

$$J(v) = \frac{1}{2}\mathcal{A}(v, v) - \mathcal{F}(v).$$

(d) Introduce the Galerkin method for the solution of (4), discuss its well-posedness, and give its algebric formulation. State and prove the quasi-optimality property (Cèa lemma) of the Galerkin method in the norm of V. Explain under which conditions this result implies the convergence of the Galerkin method.

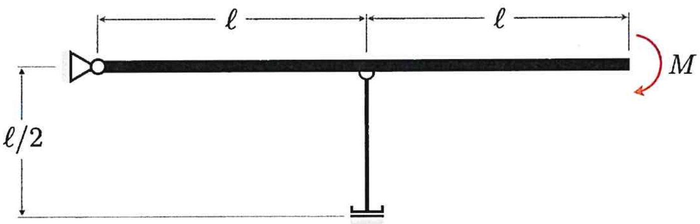
(e) Exemplify the Galerkin method with a finite element method, possibly introducing extra assumptions on the domain Ω. Comment on the speciality of its algebric form. Describe which (if any) numerical difficulties arise in the case $a \ll c$. Do you expect the convergence properties discussed in point (d) above to hold in this case as well?

Continuum Mechanics

16. An incompressible neo-Hookean cylindrical body of radius r is subject on the lateral surface to reference contact forces $-\sigma\mathbf{n}$, where $\mathbf{n}$ is the outward unit normal. Neglecting body forces, determine the value of σ such that the body is deformed into a cylinder of radius $r/2$. What are the contact forces acting on the body in the deformed configuration? *Hint*: the constitutive law for an incompressible neo-Hookean solid is $\mathbf{T} = -\pi\mathbf{I} + \mu\,\mathbf{F}\mathbf{F}^{\mathsf{T}}$, with $\mathbf{T}$ the Cauchy stress tensor, $\mathbf{F}$ the deformation gradient, and μ the shear modulus.

17. Consider the deformation $y : \mathcal{B} \to \mathcal{E}$ defined in Cartesian components by $y_1 = x_1 + \alpha\,x_2$, $y_2 = \beta\,x_2$, and $y_3 = x_3$, where α and β are positive scalars and $\{x_1, x_2, x_3\}$ are the coordinates of a material point $\mathbf{x} \in \mathcal{B}$. Compute the right Cauchy–Green strain tensor and the stretch $\lambda(\sigma) : [0,1] \to \mathbb{R}^+$ of the material curve $c(\sigma) : [0,1] \to \mathcal{E}$ with $x_1 = \sigma$, $x_2 = (\sigma - 1)^2$, and $x_3 = 0$.

18. Compute the critical torque M_c for the elastic system shown in the figure. Assume that the horizontal bar is rigid and that the vertical elastic rod is of constant bending stiffness B.



19. Two spheres of radii r_1 and r_2 are connected at their center by a linear elastic spring of stiffness k and length ℓ, much greater than the spheres' radii. The system is neutrally buoyant in a Newtonian fluid of viscosity μ. Assume that the spring is stretched by an amount δ and then released from rest at time $t = 0$. Determine the motion of the system by assuming Stokes flow for the surrounding fluid and neglecting hydrodynamic interactions between the spheres and their inertia. Comment about the cases in which: i) $r_1 = r_2$ and ii) $r_1 \gg r_2$. *Hint*: from Stokes formula, the viscous drag on a sphere moving at velocity $\mathbf{v}$ is $\mathbf{f}_{\text{drag}} = -6\pi\mu r\mathbf{v}$.

20. A linear elastic, isotropic and homogeneous cylindrical body of length ℓ and radius r is twisted about its axis through the relative rotation of the bases by the angle φ. Failure of the body occurs as φ attains the critical value of φ_f. Determine the strength of the material σ_f. *Hint*: the angle of twist per unit length is $T/(\mu I_o)$, where T is the torque magnitude, I_o is the polar moment of inertia, and μ is the shear modulus.