EXAMINATION TEST FOR SCHOLARSHIPS SUPPORTING THE PROGRAMME "LAUREA MAGISTRALE IN MATEMATICA", MAY 30, 2024

SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI (SISSA), TRIESTE
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The candidate is required to solve five of the following exercises, choosing at least one exercise in group A (exercises 1-5) and at least one in group B (exercises 6-11). The candidate must clearly indicate on the first page of the document which exercises have been chosen for the evaluation.

Group A

Exercise 1. Let $A \in M_n(\mathbb{C})$ be a diagonalisable matrix.

a) Prove the identity: $\det(e^A) = e^{\operatorname{tr} A}$.
b) Extend if possible the identity to invertible matrices A.
c) Extend if possible to the identity $\det(B) = e^{\operatorname{tr}\log(B)}$ for any matrix B in $M_n(\mathbb{C})$.

Exercise 2. Classify all minimal compact differentiable surfaces and provide a proof.

Exercise 3. A chemist has two perfectly spherical flasks B_{r_1} and B_{r_2} of radii $r_1 = 1$ and $r_2 = 2$ respectively. He needs to design two flasks that contain the same combined amount of liquid of B_{r_1} and B_{r_2}, but with different radii. The new radii still have to be rational and positive. Discuss an algorithmic procedure to find a solution in as much detail as possible. The actual solution is a fraction with a large number of digits, the computation of which is not required here.

Exercise 4. You need to hang a canvas to the wall. There is a closed string attached at the top of the canvas. The string is as long as needed, and has to go around n nails on the wall in such a way that removing any of the nails would make the canvas fall.

a) Write the problem in proper topological terms.
b) Solve it for $n = 2$.
c) Generalise the solution to arbitrary integers $n \geqslant 3$.

Exercise 5. Let μ be a positive measure with support on $\mathbb{R}$, such that the quantities

$$\mu_j := \int x^j \, d\mu(x), \quad j \in \mathbb{N},$$

exist finite for every $j \geqslant 0$ integer. Moreover, assume that the determinants

$$\begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_n & \cdots & \mu_{2n} \end{vmatrix} \neq 0 \quad \text{for every } n \geqslant 0.$$

Let $p_n(x)$ be the polynomial of degree $n \geqslant 0$ defined by the following determinant

$$p_n(x) := \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}.$$

Prove that

$$\int p_n(x) p_k(x) d\mu(x) = \begin{cases} 0, & \text{if } k \neq n, \\ \neq 0, & \text{if } k = n. \end{cases}$$

Hint: Show that it suffices to prove the above for $k \leqslant n$. Moreover, it is convenient to first *prove* that the statement is equivalent to $\int p_n(x)x^k d\mu(x) = \begin{cases} 0, & \text{if } k \leqslant n-1, \\ \neq 0, & \text{if } k = n. \end{cases}$

Group B

Exercise 6. Find

$$\sum_{n=1}^{+\infty} \int_0^1 x^n (1-x) e^x \, dx$$

.

Exercise 7. Consider the differential equation $\Delta u + u^3 = 0$, where $\Delta u = \operatorname{div}(\nabla u) = \sum_{i=1}^{4} \frac{\partial^2 u}{\partial x_i^2}$. We look for solution $u : D \to \mathbb{R}$ with $D \subset \mathbb{R}^4$. Assume that u satisfies $u(x_1, x_2, x_3, x_4) = v\left(\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}\right)$, where $v : I \to \mathbb{R}$ with $I \subset \mathbb{R}$. Which second order ODE does $v = v(r)$ solve?
Define $x(t) = v(e^t)e^t$ and $y(t) = x'(t)$. Which system

$$x' = p(x, y), \qquad y' = q(x, y)$$

does the function $z = (x, y)$ solve? Find the non constant function H such that $\frac{d}{dt} H(x(t), y(t)) = 0$ for every solution of the so-found system.

Exercise 8. A solid is described by the set $S = \{(x, y, z) \in \mathbb{R}^3 \,:\, x^2 + y^2 \leqslant 1\,, z \in [0, 3]\}$ and its density is described by a C^∞ function $\mu : S \to \mathbb{R}$ such that $\mu(x, y, z) = \rho\left(\sqrt{x^2 + y^2}\right)$ and $\mu(0, 0, 0) = 4$. The function $\rho = \rho(r)$ solve the differential equation $\rho'' - \rho = 2r$. Find the mass of the solid.

Exercise 9. Consider the the differential equation

$$D(Q) = \lambda Q, \quad \lambda \in \mathbb{C},$$

for the unknown function $Q = Q(x)$, $x \in \mathbb{C}$, where

$$D(Q) := \frac{d^2 Q}{dx^2} - 2(x^3 + bx)\frac{dQ}{dx} + \left(2Mx^2 - b - \frac{(2-M)(3-M)}{x^2}\right) Q.$$

Here $b, M \in \mathbb{C}$ are parameters. Prove that the equation has a solution $Q(x) = (c_1 + c_2 x^{-2})x^M$ if and only if λ is an eigenvalue of the matrix

$$\begin{pmatrix} -2(M+1)b & 4 \\ 2(2M-3) & b(3-2M) \end{pmatrix},$$

and $(c_1, c_2)^T$ is the corresponding eigenvector.
[Recall that $x^M := \exp(M \ln x)$ satisfies the differentiation rule $\frac{d}{dx} x^M = M x^{M-1}$.]

Exercise 10. For any $x \in [0, 1]$, let $\{x_j\}_{j \in \mathbb{N}}$ be the sequence of digits of its decimal representation, namely

$$x = 0.\, x_1 x_2 x_3 \cdots,$$

with $x_j \in \{0, 1, \ldots, 9\}$ for any j. Prove that the function $f : [0, 1] \to \mathbb{R}$ defined as

$$f(x) := \max\{x_j \mid j \in \mathbb{N}\}$$

is almost everywhere constant (with respect to Lebesgue measure).