The candidate is required to solve five of the following exercises, choosing at least one exercise in group A (exercises 1-5) and at least one in group B (exercises 6-11). The candidate must clearly indicate on the first page of the document which exercises have been chosen for the evaluation.

**Group A**

**Exercise 1.** Let $A \in M_n(\mathbb{C})$ be a diagonalisable matrix.

a) Prove the identity: $\det(e^A) = e^{\text{tr}A}$.

b) Extend if possible the identity to invertible matrices $A$.

c) Extend if possible to the identity $\det(B) = e^{\text{tr} \log(B)}$ for any matrix $B$ in $M_n(\mathbb{C})$.

**Exercise 2.** Classify all minimal compact differentiable surfaces and provide a proof.

**Exercise 3.** A chemist has two perfectly spherical flasks $B_{r_1}$ and $B_{r_2}$ of radii $r_1 = 1$ and $r_2 = 2$ respectively. He needs to design two flasks that contain the same combined amount of liquid of $B_{r_1}$ and $B_{r_2}$, but with different radii. The new radii still have to be rational and positive. Discuss an algorithmic procedure to find a solution in as much detail as possible. The actual solution is a fraction with a large number of digits, the computation of which is not required here.

**Exercise 4.** You need to hang a canvas to the wall. There is a closed string attached at the top of the canvas. The string is as long as needed, and has to go around $n$ nails on the wall in such a way that removing any of the nails would make the canvas fall.

a) Write the problem in proper topological terms.

b) Solve it for $n = 2$.

c) Generalise the solution to arbitrary integers $n \geq 3$.

**Exercise 5.** Let $\mu$ be a positive measure with support on $\mathbb{R}$, such that the quantities

$$\mu_j := \int x^j \, d\mu(x), \quad j \in \mathbb{N},$$

exist finite for every $j \geq 0$ integer. Moreover, assume that the determinants

$$\begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_n & \mu_n & \cdots & \mu_{2n} \end{vmatrix} \neq 0 \quad \text{for every } n \geq 0.$$

Let $p_n(x)$ be the polynomial of degree $n \geq 0$ defined by the following determinant

$$p_n(x) := \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}$$

Prove that

$$\int p_n(x)p_k(x) \, d\mu(x) = \begin{cases} 0, & \text{if } k \neq n, \\ \neq 0, & \text{if } k = n. \end{cases}$$
is almost everywhere constant (with respect to Lebesgue measure).

Exercise 10. 

Recall that

Here for the unknown function

Consider the differential equation

Find

Here

Exercise 9.

Let

Exercise 8.

Moreover, it is convenient to first prove that the statement is equivalent to

Exercise 7. 

Hint: Show that it suffices to prove the above for \(k \leq n\). Consider the statement is equivalent to

\[
\int p_n(x)x^k d\mu(x) = \begin{cases}
0, & \text{if } k \leq n - 1, \\
\neq 0, & \text{if } k = n.
\end{cases}
\]

Exercise 6. Find 

\[
\sum_{n=1}^{\infty} \int_0^1 x^n(1-x)e^x \, dx
\]

Exercise 7. Consider the differential equation \(\Delta u + u^3 = 0\), where \(\Delta u = \text{div}(\nabla u) = \sum_{i=1}^{4} \frac{\partial^2 u}{\partial x_i^2}\). We look for solution \(u : D \to \mathbb{R}\) with \(D \subseteq \mathbb{R}^4\). Assume that \(u\) satisfies \(u(x_1, x_2, x_3, x_4) = v\left(\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}\right)\), where \(v : I \to \mathbb{R}\) with \(I \subseteq \mathbb{R}\). Which second order ODE does \(v = v(r)\) solve? Define \(x(t) = v(e^t)e^t\) and \(y(t) = x'(t)\). Which system

\[
x' = p(x, y), \quad y' = q(x, y)
\]
does the function \(z = (x, y)\) solve? Find the non constant function \(H\) such that \(\frac{\partial}{\partial t} H(x(t), y(t)) = 0\) for every solution of the so-found system.

Exercise 8. A solid is described by the set \(S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, z \in [0, 3]\}\) and its density is described by a \(C^\infty\) function \(\mu : S \to \mathbb{R}\) such that \(\mu(x, y, z) = \rho\left(\sqrt{x^2 + y^2}\right)\) and \(\mu(0,0,0) = 4\). The function \(\rho = \rho(r)\) solve the differential equation \(\rho'' - \rho = 2r\). Find the mass of the solid.

Exercise 9. Consider the the differential equation

\[
D(Q) = \lambda Q, \quad \lambda \in \mathbb{C},
\]

for the unknown function \(Q = Q(x), x \in \mathbb{C}\), where

\[
D(Q) := \frac{d^2 Q}{dx^2} - 2(\lambda + bx) \frac{dQ}{dx} + \left(2Mx^2 - b - \frac{(2-M)(3-M)}{x^2}\right)Q.
\]

Here \(b, M \in \mathbb{C}\) are parameters. Prove that the equation has a solution \(Q(x) = (c_1 + c_2x^2)x^M\) if and only if \(\lambda\) is an eigenvalue of the matrix

\[
\begin{pmatrix}
-2(M+1)b & 4 \\
2(2M-3) & b(3-2M)
\end{pmatrix},
\]

and \((c_1, c_2)^T\) is the corresponding eigenvector.

[Recall that \(x^M := \exp(M \ln x)\) satisfies the differentiation rule \(\frac{d}{dx}x^M = Mx^{M-1}\).]

Exercise 10. For any \(x \in [0, 1]\), let \(\{x_j\}_{j \in \mathbb{N}}\) be the sequence of digits of its decimal representation, namely

\[
x = 0.x_1x_2x_3\cdots,
\]

with \(x_j \in \{0, 1, \ldots, 9\}\) for any \(j\). Prove that the function \(f : [0,1] \to \mathbb{R}\) defined as

\[
f(x) := \max\{x_j \mid j \in \mathbb{N}\}
\]
is almost everywhere constant (with respect to Lebesgue measure).