Scuola Internazionale Superiore di Studi Avanzati, Trieste
Entrance examination for the grants for "Laurea Magistrale in Matematica"
Written exam: September 3, 2024

The candidate is required to solve five among the following exercise, choosing at least one exercise in group A and at least one in group B. The candidate should indicate in a clear way on the first page which are the chosen exercises that must be evaluated (in any case no more than five).

Group A

1. Consider the function defined for $x \in \mathbb{R}$ by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that it has a primitive in $\mathbb{R}$, that is there exists a differentiable function $F : \mathbb{R} \to \mathbb{R}$ such that $\frac{d}{dx}F(x) = f(x)$ for all $x \in \mathbb{R}$.

2. Let $A \subset \mathbb{R}^3$ be the 3-dimensional simplex (tetrahedron) with vertices $(0,0,0)$, $(6,2,1)$, $(3,4,1)$ and $(1,1,0)$. Compute

$$\int_A (x + y + z)\, dxdydz.$$

3. Let $R \geq 0$, $n \in \mathbb{N}$. Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, such that $|f(z)| \leq |z|^n$ for all $|z| > R$. Prove that f is a polynomial, and estimate its degree.

4. Recall that a subset S of a (topological) vector space is *convex* (resp. *strictly convex*) if for every $x, y \in S$, every point on the line segment connecting x, y other than the endpoints belongs to S (resp. to the topological interior of S).

Let $C([0,1])$ be the space of continuous real-valued functions on $[0,1]$, equipped with the uniform topology. Consider the set

$$S = \left\{ f \in C([0,1]) \, : \, \sup_{x \in [0,1]} |f(x)| \leq 1 \right\}.$$

(i) Prove that S is not strictly convex.

(ii) Is S convex?

5. Consider the ODE

$$y''' + 6y'' + 11y' + 6y = 0.$$

Prove that there are no non-trivial bounded solutions defined on $\mathbb{R}$.

Group B

6. Let $M(n,\mathbb{R})$ be the set of real $n \times n$ matrices, equipped with the Euclidean topology induced by the standard identification $M(n,\mathbb{R}) \simeq \mathbb{R}^{n^2}$. Let $O(n) \subset M(n,\mathbb{R})$ be the orthogonal group:

$$O(n) = \{M \in M(n,\mathbb{R}) \mid MM^t = I\},$$

endowed with the subset topology.

(i) Determine the number of connected components of $O(n)$.

(ii) Determine whether $O(n)$ is compact.

7. Let n be a positive natural number and V be a n-dimensional vector space over a field K ($n \geq 1$).

(i) Prove that V is finite (as a set) if and only if K is a finite field, and give an expression for the number of elements of V when this occurs.

(ii) Prove that, if K has prime order $p > 2$, there exists a non-trivial (i.e., different from the identity) endomorphism φ of V such that $\varphi^p = \varphi$ and $\varphi^m \neq \varphi$ for all $1 < m < p$.

8. Let G be the group of automorphisms of the polynomial ring $\mathbb{Z}/3\mathbb{Z}\,[x]$. Describe the elements of G and find if G is commutative.

9. Let $K \subseteq L$ be an extension of fields. Suppose that $a, b \in L$ are algebraic over K. Show that $a + b$ is also algebraic over K.

10. Let D be a unitary circle placed on a horizontal bar and P a point on it. The circle is rolling clockwise uniformly on the horizontal bar (without slipping), and the bar is uniformly lifting in the vertical direction (imagine a wheel on a lift). Fix a Cartesian coordinate system such that the bar is always perpendicular to the y-axis and such that at time $t = 0$ the point P corresponds to $(0,0)$, the contact point between the circle and the bar.

(i) As t varies, the point P on the circle moves. Provide equations for the trajectory of the point P, considering that in $t = 2\pi$ the circle has done a single rotation (so that the point P lies again on the bar) and the bar has moved ℓ upwards for some fixed $\ell \geq 0$.

(ii) Denote by $\mathcal{L}_\ell$ the length of the trajectory of P for a single rotation of the circle as $\ell \geq 0$ varies. Discuss the existence of (global) maxima and minima for $\mathcal{L}_\ell$.