Scuola Internazionale Superiore di Studi Avanzati, Trieste

Entrance examination for the grants for "Laurea Magistrale in Matematica"

Written exam: September 6, 2018

The candidate is required to solve five among the following exercise, choosing at least one exercise in group A (exercises 1-5) and at least one in group B (exercises 6-10). The candidate should indicate in a clear way on the first page which are the chosen exercises that must be evaluated (in any case no more than five).

Group A

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 and periodic function (i.e. there exists $T > 0$ such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$). Consider the set:

$$P = \{T > 0 \text{ such that } f(x+T) = f(x) \text{ for all } x \in \mathbb{R}\}.$$

(a) Prove the following statement (s) : "if f is not constant, then $\inf P > 0$".

(b) Is (s) still true if we simply assume that f is only *continuous*? Prove it or provide a counterexample.

(c) Is (s) still true if we only assume that f is *continuous at one point*? Prove it or provide a counterexample.

(d) Is (s) still true if we remove the condition that f is continuous? Prove it or provide a counterexample.

2. Let X be a subset of $\mathbb{R}^n$ with the Euclidean topology. Prove that X is compact if and only if every continuous function $f : X \to \mathbb{R}$ is bounded.

3. Let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence of real numbers defined by:

$$x_1 = 1 \quad \text{and} \quad x_{n+1} = x_n - \frac{1}{n(n+1)}.$$

Determine the expression of the general element x_n of the sequence.

4. Let $(A, \mathcal{B}, \mu)$ be a measure space with finite measure and let $f : A \to \mathbb{R}_+$ be a measurable function. Let $\lambda : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by:

$$\lambda(t) = \mu\left(\left\{x \in A \,|\, f(x) > \frac{1}{t}\right\}\right).$$

Prove that:

$$\int_A f \, d\mu = \int_0^{+\infty} \frac{\lambda(t)}{t^2} dt.$$

(Hint: consider the function $K : A \times (0, \infty) \to \mathbb{R}$ defined by $K(x,t) = \frac{1}{t^2}\chi_{\{f(x)>\frac{1}{t}\}}(x)$.)

5. Let $f : (0, +\infty) \to (-1, 0)$ be a function such that

$$\lim_{x\to+\infty} f(x) = 0.$$

Prove that f is not convex.

Group B

6. Let A be a $n \times n$ matrix with complex entries with $n \geq 2$.

(a) Prove that if A is *nilpotent* (i.e. $A^r = 0$ for some $r \in \mathbb{N}$), then each eigenvalue of A is zero. Then find the characteristic polynomial of A.

(b) More generally, decide for which values $c \in \mathbb{C}$ a matrix A such that $A^r = cI_n$ (here I_n denotes the identity matrix) is diagonalizable.

7. In the real affine plane $\mathbb{A}^2$ with coordinates $(O; x, y)$ consider the ellipse:

$$\Gamma : x^2 + 4y^2 = 4.$$

Let A, B be the points of intersection of Γ with the x-axis and P the point of intersection of Γ with the y-axis and with positive y-coordinate.

(a) Determine the coordinates of the points A, B and P and the line t tangent to Γ at P;

(b) write the equation of the sheaf $\mathcal{F}$ of conics tangent to Γ at P and passing through A and B;

(c) find the reducible conics of $\mathcal{F}$.

(d) Let $\mathcal{G}$ be the sheaf of conics which are simultaneously tangent to Γ at A and P. *Without writing the equation for* $\mathcal{G}$, determine the conics common to both $\mathcal{F}$ and $\mathcal{G}$. In general, how many conics are common to two distinct sheaves of conics?

8. Let $\mathbb{Z}_3[X]$ be the polynomial ring in the variable X and with coefficients in $\mathbb{Z}_3 := \mathbb{Z}/3\mathbb{Z}$ and consider the quotient ring $A := \mathbb{Z}_3[X]/(X^2 + \overline{2})$.

(a) Determine the number of elements of A;

(b) prove that A is not a domain;

(c) find the set $ZD(A)$ of the zero divisors of A. *(hint: look also at the next point (d))*

(d) Which nonzero elements in $A \backslash ZD(A)$ are invertible? For each of them, determine its inverse.

9. Let $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ be the real line with the standard topology $\mathcal{T}_{\mathbb{R}}$ and let $\mathcal{C} = \{A \subseteq \mathbb{R} \mid \mathbb{R} \backslash A \text{ is compact}\}$. Consider the "$n$-points-compactification" of X, i.e. the topological space $X_n = \mathbb{R} \cup \{\infty_1, \dots, \infty_n\}$ together with the topology $\mathcal{T}$ defined by:

$$\mathcal{T} = \mathcal{T}_{\mathbb{R}} \cup \left(\bigcup_{k=1}^{n} \{A \cup \{\infty_k\} \mid A \in \mathcal{C}\} \right).$$

(a) Prove that X_n is a compact topological space, non-Hausdorff for $n \geq 2$.

(b) Compute the fundamental group of X_n.

10. A point particle is constrained to move without friction along the curve C having equation

$$y = c \cosh \frac{x}{c},$$

where $c > 0$. The only active force acting on the point is its (constant) weight, having a vertical downward direction (direction of the negative y axis). At $t = 0$ the point is in the position $x = c$ and has zero speed.

(a) Write the arc length along C as a function of x.

(b) Write the equation of motion of the point in terms of the arc length as a function of time.

(c) Find an integral of the motion (conserved quantity).

(d) Compute the time after which the point stops.