Scuola Internazionale Superiore di Studi Avanzati, Trieste

Entrance examination for the grants for "Laurea Magistrale in Matematica"

Written exam: September 5, 2019

The candidate is required to solve up to five among the following exercises, choosing at least one exercise in group A (exercises 1-5) and at least one in group B (exercises 6-10). The candidate should indicate in a clear way on the first page which are the chosen exercises that must be evaluated (in any case no more than five).

Group A

1.

(a) Let $(a,b) \subset \mathbb{R}$ be an open bounded interval and $f : (a,b) \to \mathbb{R}$ a uniformly continuous function. Prove that f can be extended to a continuous function $\hat{f} : [a,b] \to \mathbb{R}$.

(b) More generally, let (X_1, d_1) be a compact metric space and (X_2, d_2) be a complete metric space. Prove that if $D \subseteq X_1$ is dense and $f : D \to X_2$ is uniformly continuous, then f can be extended to a continuous function $\hat{f} : X_1 \to X_2$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous and periodic function, i.e. there exists $T > 0$ such that for all $x \in \mathbb{R}$ we have $f(x+T) = f(x)$. Prove that f is uniformly continuous.

2. Let $(a,b) \subset \mathbb{R}$ be an open interval and $f : (a,b) \to \mathbb{R}$ a convex function.

(a) Prove that f is continuous.

(b) If now $[a,b]$ is a closed interval and $f : [a,b] \to \mathbb{R}$ is convex, is it necessarily continuous?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, locally Lipschitz function, and consider the autonomous differential equation

$$u' = f(u). \tag{1}$$

Let $u : [0,\infty) \to \mathbb{R}$ be a solution to (1) such that the limit $\lim_{t\to\infty}(u(t))$ exists and equals $u_\infty \in \mathbb{R}$. Prove that $f(u_\infty) = 0$.

4.

(a) Prove that each compact subspace of a Hausdorff topological space is closed.

(b) Let $f : X \to Y$ be a continuous and bijective function between topological spaces. Suppose that X is compact and Y is Hausdorff and prove that f is a homeomorphism.

(c) Consider the interval $[0, 2\pi) \subset \mathbb{R}$ and $S^1 = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$, each endowed with the subspace topology induced by the standard one of $\mathbb{R}^n$. Prove that the function $f : [0, 2\pi) \to S^1$ with $f(t) = (\cos t, \sin t)$ is a bijective continuous function that is not a homeomorphism.

5.

(a) Let $z_0 \in \mathbb{C}$ and $g : \mathbb{C}\backslash\{z_0\} \to \mathbb{C}$ be a holomorphic function. Assume that z_0 is a pole of g. Prove that the function $g'(z)/g(z)$ has a pole of order 1 in z_0.

(b) Let now $f : \mathbb{C}\backslash\{z_0\} \to \mathbb{C}$ be a holomorphic function. Prove that z_0 cannot be a pole of the function $e^{f(z)}$.

Group B

6. A point particle of mass m is bound to move on a circumference of radius R, which is rotating around a vertical diameter with angular speed $\omega(t) = \alpha t$, where α is constant. The only active force acting on the particle is weight, pointing along the downward vertical; the gravity field $\boldsymbol{g}$ is assumed to be constant. The circumference exerts no friction on the particle.

(a) Write the equation of motion of the particle.

(b) Assuming that at $t = 0$ the particle is in the lowest position with zero speed, show that it keeps still for all $t > 0$.

(c) Assuming that at $t = 0$ the particle is in the lowest position with speed v, compute the force exerted by the circle on the particle at that time.

7. Let $M_n(\mathbb{C})$ be the space of $n \times n$ matrices with complex entries, topologized by identifying it with $\mathbb{C}^{n^2}$ with the Euclidean topology.

(a) Let $D(n) \subset M_n(\mathbb{C})$ be the subspace of diagonalizable matrices. Show that $D(n)$ is dense in $M_n(\mathbb{C})$.

(b) For every $A \in M_n(\mathbb{C})$ let

$$\exp A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Show that exp is a well-defined map $M_n(\mathbb{C}) \to M_n(\mathbb{C})$, and that it is continuous.

(c) Prove that

$$\det \exp A = e^{\operatorname{tr} A}.$$

(d) Let $GL_n(\mathbb{C}) \subset M_n(\mathbb{C})$ by the subspace of invertible matrices. Shows that exp takes values in $GL_n(\mathbb{C})$.

8.

(a) Let G be a group and $g \in G$ be an element of order n; compute the order of g^k for each $k \in \mathbb{N}$.

(b) Let C be a finite cyclic group of order n. Prove that for each natural number $d \geq 1$ dividing n there exists a subgroup of C of order d.

(c) Prove that $\mathbb{A}_4$, the alternating group of 4 letters, does not contain any subgroup of order 6.

9.

(a) Prove that $\mathbb{R}$ and $\mathbb{R}^n$ $(n > 1)$ with the standard topologies are not homeomorphic.

(b) Prove that $\mathbb{R}^2$ and $\mathbb{R}^3$ with the standard topologies are not homeomorphic.

10. Let $\sigma : \{1, \ldots, n^2\} \to \{1, \ldots, n^2\}$ be a permutation. Consider the vector space $M_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$ of matrices with complex entries and the endomorphism

$$L_\sigma : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

that permutes the entries of a matrix according to the permutation σ. Prove that L_σ is diagonalizable.