Scuola Internazionale Superiore di Studi Avanzati, Trieste

Area of Mathematics

Selection for the PhD Course in Geometry and Mathematical Physics

Written test, September 5, 2016

Each candidate is required to solve at least one of the following exercises. Every answer must be sufficiently motivated.

A. CLASSICAL MECHANICS

Consider the Hamiltonian

$$H(q_1, p_1, p_2, q_2) = \frac{1}{2}p_1^2 + \frac{1}{2}q_2^2 p_2^2 + e^{-q_1 q_2 p_2}.$$

Here (q_1, q_2) and (p_1, p_2) respectively denote Cartesian coordinates and the canonically conjugate momenta.

(**1**) Find two one-parameter families of symmetries and the associated conserved quantities.

(**2**) Qualitatively describe the motion in the phase space.

(**3**) Compute *explicitly* $q_1 = q_1(t)$ and $p_1 = p_1(t)$ as functions of time and initial conditions.

B. QUANTUM MECHANICS

Consider a system of three spinless particles, each of mass m, moving on a straight line and bound to each other by two-body harmonic forces. The Hamiltonian of the system is

$$H \;=\; -\frac{\hbar^2}{2m}\sum_{j=1}^{3}\frac{\partial^2}{\partial x_j^2} + \frac{k}{2}\sum_{1\leqslant j<\ell\leqslant 3}(x_j - x_\ell)^2$$

($x_1, x_2, x_3 \in \mathbb{R}$, $m, k > 0$) acting on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3)$ and defined on its natural domain of self-adjointness.

(**1**) Find a *linear* change of coordinates $(x_1, x_2, x_3) \xmapsto{\mathcal{R}} (y_1, y_2, y_3)$ where, say, y_3 is the coordinate of the centre of mass, such that

- $\mathcal{R}$ induces a *unitary* transformation $U_{\mathcal{R}} : L^2(\mathbb{R}^3, \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3) \xmapsto{\cong} L^2(\mathbb{R}^3, \mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3)$,
- the transformed Hamiltonian $U_{\mathcal{R}} H U_{\mathcal{R}}^{-1}$ is the sum of three independent one-body Hamiltonians.

(**2**) Find the spectrum of H (apart from the degeneracy due to the translation invariance).

(**3**) If the three particles were indistinguishable bosons, would the *absolute* ground state be also *bosonic*? (bosonic = symmetric under variable permutations)

C. DIFFERENTIAL EQUATIONS

Given n functions $a_1(z), \ldots, a_n(z)$ of the complex variable $z = x + iy$ consider the linear differential equation

$$\frac{\partial^n f}{\partial z^n} + a_1(z)\frac{\partial^{n-1} f}{\partial z^{n-1}} + \cdots + a_n(z) f = 0 \tag{1}$$

for a complex-valued function $f = f(z)$. Here the complex derivative is defined by the usual formula

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right).$$

Denote by V the set of *entire* functions $f(z)$ satisfying the differential equation (1). (Recall that a function $f(z)$ of complex variable z is called entire if it is a holomorphic function on the entire complex plane $z \in \mathbb{C}$.)

(**1**) Prove that V is a complex linear space of complex dimension $\dim_{\mathbb{C}} V$ less or equal than n.

(**2**) Let $a_1(z)$ be an entire function. Prove that the complex dimension of V is equal to n if and only if all the remaining coefficients $a_2(z), \ldots, a_n(z)$ are also entire functions.

(**3**) Find an example of a differential equation of the form (1) with the coefficient $a_1(z)$ being an entire function but

$$0 < \dim_{\mathbb{C}} V < n.$$

D. COMPLEX ANALYSIS

Let C be the unit circle $C : \{|z| = 1\}$. Suppose f is analytic on an open set containing the unit disk and $|f(z)| < 1$ on C. Show that f has exactly one fixed point inside the unit disk, namely there is one and only one z in the unit disk for which $f(z) = z$.

Hint: We remind that if a function $H(z)$ is analytic on a domain containing the disk, the number of solutions of the equation $H(z) = 0$ (counted with multiplicity) is given by the *index* formula:

$$\oint_C \frac{H'(z)dz}{H(z)2i\pi}$$

E. COMPLEX GEOMETRY

Let X be a complex manifold of complex dimension n. Let $z = (z_1, \dots, z_n)$ denote local holomorphic coordinates on X.

(a) Prove that the operators ∂ and $\bar{\partial}$, mapping smooth functions on X to 1-forms, defined locally by
$$\partial f = \sum_i \frac{\partial f}{\partial z_i} dz_i, \quad \bar{\partial} f = \sum_i \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$$
are in fact globally defined, and that one has $\partial^2 = \bar{\partial}^2 = 0$ and a decomposition for the exterior differential $d = \partial + \bar{\partial}$.

Let $T^{1,0}X$ denote the holomorphic tangent bundle. Recall a Hermitian metric g on X is a smoothly varying Hermitian inner product on the fibres of $T^{1,0}X$. Locally it is simply a tensor
$$g = \sum_{i,j} g_{i\bar{j}}(z)(dz_i \otimes d\bar{z}_j + d\bar{z}_j \otimes dz_i).$$
where $g_{i\bar{j}}(z)$ is a smooth function taking values in positive definite Hermitian $n \times n$ matrices. Its associated metric form and Ricci form are the differential 2-forms defined respectively by
$$\omega_g = \sqrt{-1} \sum_{i,j} g_{i\bar{j}}(z) dz_i \wedge d\bar{z}_j, \quad \mathrm{Ric}(g) = -\sqrt{-1}\partial\bar{\partial} \log \det(g_{i\bar{j}}(z)).$$

(b) Express the closedness condition $d\omega_g = 0$ in local coordinates.

(c) Prove that $\mathrm{Ric}(g)$ is a globally defined 2-form, that it is closed, and that the corresponding de Rham cohomology class $[\mathrm{Ric}(g)]$ is independent of the choice of g. (When $d\omega_g = 0$, $\mathrm{Ric}(g)$ is in fact the same as the Ricci curvature of g, hence the name. This fact is *not* needed anywhere in the present exercise, although the candidates can use it if they think it may be helpful.)

(d) Let $X = \{z \in \mathbb{C}^2 : |z| < 1\}$, choose
$$g_{i\bar{j}} = \partial_{z_i} \bar{\partial}_{\bar{z}_j} \log(1 - |z_1|^2 - |z_2|^2).$$
Compute ω_g, $d\omega_g$, and $\mathrm{Ric}(g)$. Is there a simple relation between ω_g and $\mathrm{Ric}(g)$?

(e) Let $X = \mathbb{CP}^2$ with homogeneous coordinates x_0, x_1, x_2, choose
$$g_{i\bar{j}} = \partial_{z_i} \bar{\partial}_{\bar{z}_j} \log(1 + |z_1|^2 + |z_2|^2)$$
where $z_k = \frac{x_k}{x_0}$, $k = 1, 2$ are coordinates on the patch $x_0 \neq 0$. Show that g extends to a Hermitian metric g_{FS} on all X and compute $\omega_{g_{FS}}$, $d\omega_{g_{FS}}$ and $\mathrm{Ric}(g_{FS})$. Is there a simple relation between $\omega_{g_{FS}}$ and $\mathrm{Ric}(g_{FS})$?

(f) Fix a positive integer d and let $X = \{x_0^d + x_1^d + x_2^d = 0\} \subset \mathbb{CP}^2$. Let $i_X : X \hookrightarrow \mathbb{CP}^2$ denote the inclusion. Show that we have

$$\mathrm{Ric}(i_X^* g_{FS}) = (3-d)\, i_X^* \omega_{g_{FS}} + \text{an } \textit{explicit} \text{ exact 2-form on all } \mathbb{CP}^2.$$

(g) Let X be as in the previous item. Compute $\int_X \mathrm{Ric}(i_X^* g_{FS})$.

F. ALGEBRAIC GEOMETRY

We will work over an algebraically closed field K; you may assume that K is the complex numbers, or work with complex manifolds instead of varieties. You can skip all or part of one or more steps and use the statements in subsequent steps.

(**1**) (Segre embedding) Fix positive integers n and m, and let $N := (n+1)(m+1) - 1$. We view $\mathbf{P}^N$ as the projectivization of the space of $(n+1) \times (m+1)$ matrices (i.e., the quotient of the open locus $K^{N+1} \setminus \{0\}$ of nonzero matrices by the action of K^* via scalar multiplication). Let $X \subset \mathbf{P}^N$ be the locus of rank one matrices. Show that X is a closed subvariety (or complex holomorphic submanifold) isomorphic to $\mathbf{P}^n \times \mathbf{P}^m$.

(**2**) From now on, assume that $n = m = 2$ and hence $N = 8$. We say that a line L in $\mathbf{P}^8$ is *secant* to X if $L \cap X$ contains at least two points. Let L be a secant line, and $p \in L \setminus X$; show that the rank of the (3×3) matrix corresponding to p is 2. Conversely, let $p \in \mathbf{P}^8$ be a point corresponding to a rank 2 matrix; show that there is a secant line L such that $p \in L$.

(**3**) Let $Y \subset \mathbf{P}^8$ be the locus of matrices of rank at most 2. Show that Y is a cubic (degree 3) hypersurface. Show that Y is the *secant variety* of X, i.e., the closure in $\mathbf{P}^8$ of the union of all secant lines. Deduce that it also contains all tangent lines to X.

(**4**) Show that Y is irreducible and that its singular locus is exactly X (in complex geometry, show that $Y \setminus X$ is a connected complex manifold).

(**5**) Fix $p \in \mathbf{P}^8$, and let $H \subset \mathbf{P}^8$ a hyperplane (i.e., codimension 1 linear projective subspace) which does not contain p. Let $\pi : \mathbf{P}^8 \setminus \{p\} \to H$ be the projection, defined by $\pi(q) := L_q \cap H$ where L_q is the unique line that contains p and q. Let $q_1, q_2 \in \mathbf{P}^8 \setminus \{p\}$; show that $\pi(q_1) = \pi(q_2)$ if and only if p, q_1 and q_2 are on a line.

(**6**) Assume now that p is not in Y; in particular, this implies that $X \subset \mathbf{P}^8 \setminus \{p\}$. Let $\bar{X} \subset H$ be the image of X via π. Show that $\pi|_X : X \to \bar{X}$ is bijective. Show that $\bar{X}$ is closed in H (hint: you need to use that X is proper/complete).

(**7**) Show that for every $q \in X$, the differential of $\pi|_X$ in the point q is injective. Hint: the line L_q is not tangent to X at q. (Optional) Deduce that $\pi|_X : X \to \bar{X}$ is an isomorphism.
