The candidate is required to solve some of the following exercises. A mark sufficient to be admitted to the oral exam is attained by correctly and completely solving at least one exercise. Every answer must be motivated.

A. CLASSICAL MECHANICS

We study the motion of a particle with cartesian coordinates $\vec{x} = (x_1, x_2, x_3)$ and conjugate canonical momenta $\vec{p} = (p_1, p_2, p_3)$. Denote by $\vec{L} = (L_1, L_2, L_3)$ the angular momentum $\vec{L} = \vec{x} \times \vec{p}$. Here $\times$ is the cross product (or vector product). It may be sometimes convenient to denote the components of $\vec{L}$ as entries of a skew-symmetric matrix $L_{ij} := x_i p_j - x_j p_i$, $1 \leq i, j \leq 3$.

1) Consider the Hamiltonian

$$H := \frac{L_1^2 + L_2^2 + L_3^2}{2(x_1^2 + x_2^2 + x_3^2)} + \frac{1}{2}(k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2),$$

where $(k_1, k_2, k_3) \in \mathbb{R}^3$.

1.a) Prove that the motion is constrained to a sphere $x_1^2 + x_2^2 + x_3^2 = r^2$, where $r > 0$ is a constant (depending on initial conditions).

1.b) Let $k_1 = k_2 = k_3$. Explicitly compute $\vec{L} = \vec{L}(t)$ as function of time $t$.

2) Consider now the class of Hamiltonians

$$\tilde{H} = \tilde{H}(L_1, L_2, L_3),$$

which only depend on $\vec{L}$. Prove that there exists a function $\Psi(L_1, L_2, L_2)$ which is a first integral (namely, a constant of the motion) for all the Hamiltonians (2). Explicitly find the function $\Psi(L_1, L_2, L_2)$.

3) Then, consider the specific case when (2) is as follows:

$$\tilde{H} = \frac{L_1^2}{2A} + \frac{L_2^2}{2B} + \frac{L_3^2}{2C}, \quad 0 < A \leq B \leq C.$$
3.a) Qualitatively study the motion of $\vec{L}$ in $\mathbb{R}^3$, identified with the space of the components $(L_1, L_2, L_3)$. Geometrically characterise the orbit of $(L_1, L_2, L_3)$ in $\mathbb{R}^3$. The candidate may draw a figure of the orbit.

3.b) Write the Hamilton equations of motion of $\vec{L}$.
   
   In case $A = B$, solve them and compute $L_1 = L_1(t)$, $L_2 = L_2(t)$, $L_3 = L_3(t)$ explicitly as functions of time $t$.

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B. QUANTUM MECHANICS

Consider a system of two spin-1/2 particles, interacting with an external magnetic field along the z axis and among themselves. The Hamiltonian is

$$H = 4a\vec{s}_1 \cdot \vec{s}_2 + 2b(s_{1z} + s_{2z}).$$  \hspace{1cm} (3)

Taking $\hbar = 1$ for simplicity, represent as usual the two spins as

$$s_{1i} = \frac{1}{2}\sigma_i \otimes 1_2, \quad s_{2i} = \frac{1}{2}1_2 \otimes \sigma_i,$$ \hspace{1cm} (4)

where $\sigma_i$ are the Pauli matrices and $1_2$ is the $2 \times 2$ identity.

a) At time $t = 0$, we measure both spins in the $\hat{x}$ direction and we get $1/2$ for both. Write down the corresponding state $\psi_0$.

b) Write down the matrix form of $H$, specifying what basis states you are using.

c) Write down the projectors $P_s$ on the singlet and $P_t$ on the space of triplet states. Show that $P_s$ commutes with $H$; in other words, $H$ doesn’t mix the singlet $|s\rangle$ with the triplet. Evolving $\psi_0$, what is the probability that at time $t$ we measure the state to be $|s\rangle$?

d) Evolving $\psi_0$, at a time $t$ we measure both spins along $\hat{z}$. What is the probability of finding $1/2$?

e) Suppose now we evolve $\psi_0$, and at a time $T$ we measure the first spin along $\hat{z}$, finding $1/2$. We then evolve the resulting state for a further time $T$, and we measure the second spin along $\hat{z}$. What is the probability of finding $1/2$?
C. COMPLEX ANALYSIS

a) Consider the sequence of polynomials \( p_n(z) := z^n + 2z^3 + 2z + 4 \).
Show that any sequence of mutually distinct roots of the collection \( \{ p_n : n \in \mathbb{N} \} \) converges in modulus to the unit circle \(|z| = 1|\).

b) Find all functions \( f(z) \) satisfying:
1. \( f(z) \) is analytic on \( \{ \Im(z) > 0 \} \);
2. \( f(z) \) is continuous on \( \{ \Im(z) \geq 0 \} \);
3. \( f(z) \) is real–valued on the real axis;
4. \( |f(z)| > |\sin(z)| \) on \( \{ \Im(z) > 0 \} \).

D. DIFFERENTIAL GEOMETRY

Fix \( r > 0 \) and let \( S = \{ x^2 + y^2 + z^2 = r^2 \} \subset \mathbb{R}^3 \) denote the sphere of radius \( r \) endowed with the metric induced from \( \mathbb{R}^3 \).

(a) Compute the geodesic curvature \( k_g(h) \) of the curve \( \gamma_h \) on \( S \) given by the equation \( z = h \) for \( 0 \leq h < r \). Is \( k_g(h) \) bounded? Is its ratio with the curvature of \( \gamma_h \) thought of as a curve in \( \mathbb{R}^3 \) bounded?

(b) Suppose \( \gamma_h(s) \) is a simple parametrisation by arc length of \( \gamma_h \) and let \( X(s) \) denote a parallel unit vector field along \( \gamma_h(s) \). Show that the infinitesimal change, with respect to \( s \), of the angle formed by \( X(s) \) and the curve \( \gamma_h(s) \) equals the geodesic curvature \( k_g(s) \).

(c) Let \( 0 \leq \theta(h) \leq \pi \) denote the convex angle formed by \( X(0) \) and its parallel transport along \( \gamma_h(s) \) after a time equal to the length of \( \gamma_h(s) \). Show that \( \theta(h) \) is well defined and achieves its maximum \( \theta_M \) for a unique \( \bar{h} \in (0, r) \). Compute \( \theta_M \) and \( \bar{h} \).

E. ALGEBRAIC GEOMETRY

Let \( k \) be an algebraically closed field.

a) Given a polynomial \( f \in k[x_1, \ldots, x_n] \), prove that it determines a homogeneous polynomial \( f^{\text{hom}} \in k[x_0, x_1, \ldots, x_n] \).
If $I \subset k[x_1, \ldots, x_n]$ is an ideal, let $I^{\text{hom}}$ be the ideal of $k[x_0, x_1, \ldots, x_n]$ generated by the set

$$\{f^{\text{hom}} \mid f \in I\}.$$  

b) Let $X \subset \mathbb{A}_k^n$ be an affine variety determined by an ideal $I \subset k[x_1, \ldots, x_n]$. Prove that the closure $\bar{X}$ of $X$ in $\mathbb{P}_k^n$ is the projective variety $V(I^{\text{hom}})$.

c) Let $X \subset \mathbb{A}_k^2$ be the affine variety determined by the ideal $I = (f, g) \subset k[x, y]$ with $f(x, y) = x, \ g(x, y) = y - x^2$. Determine its closure $\bar{X}$ in $\mathbb{P}_k^2$.

d) Compare $\bar{X}$ with the projective variety $V(f^{\text{hom}}, g^{\text{hom}})$.

e) Let $f_1, \ldots, f_m$ be generators of an ideal $I \subset k[x_1, \ldots, x_n]$. Find a condition under which $V(f_1^{\text{hom}}, \ldots, f_m^{\text{hom}}) = V(I^{\text{hom}})$. 

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