Scuola Internazionale Superiore di Studi Avanzati, Trieste

Area of Mathematics

Selection for the PhD Course in Geometry and Mathematical Physics

Written test, September 11, 2019

The candidate is required to solve some of the following exercises. A mark sufficient to be admitted to the oral exam is attained by correctly and completely solving at least one exercise. Every answer must be motivated.

A. CLASSICAL MECHANICS

We study the motion of a particle with cartesian coordinates $\vec{x} = (x_1, x_2, x_3)$ and conjugate canonical momenta $\vec{p} = (p_1, p_2, p_3)$. Denote by $\vec{L} = (L_1, L_2, L_3)$ the angular momentum $\vec{L} = \vec{x} \times \vec{p}$. Here $\times$ is the cross product (or vector product). It may be sometimes convenient to denote the components of $\vec{L}$ as entries of a skew-symmetric matrix $L_{ij} := x_i p_j - x_j p_i$, $1 \leq i, j \leq 3$.

1) Consider the Hamiltonian

$$H := \frac{L_1^2 + L_2^2 + L_3^2}{2(x_1^2 + x_2^2 + x_3^2)} + \frac{1}{2}(k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2), \tag{1}$$

where $(k_1, k_2, k_3) \in \mathbb{R}^3$.

1.a) Prove that the motion is constrained to a sphere $x_1^2 + x_2^2 + x_3^2 = r^2$, where $r > 0$ is a constant (depending on initial conditions).

1.b) Let $k_1 = k_2 = k_3$. Explicitly compute $\vec{L} = \vec{L}(t)$ as function of time t.

2) Consider now the class of Hamiltonians

$$\widetilde{H} = \widetilde{H}(L_1, L_2, L_3), \tag{2}$$

which only depend on $\vec{L}$. Prove that there exists a function $\Psi(L_1, L_2, L_2)$ which is a first integral (namely, a constants of the motion) for *all* the Hamiltonians (2). Explicitly find the function $\Psi(L_1, L_2, L_2)$.

3) Then, consider the specific case when (2) is as follows:

$$\widetilde{H} = \frac{L_1^2}{2A} + \frac{L_2^2}{2B} + \frac{L_3^2}{2C}, \quad 0 < A \leq B \leq C.$$

3.a) Qualitatively study the motion of $\vec{L}$ in $\mathbb{R}^3$, identified with the space of the components (L_1, L_2, L_3). Geometrically characterise the orbit of (L_1, L_2, L_3) in $\mathbb{R}^3$. The candidate may draw a figure of the orbit.

3.b) Write the Hamilton equations of motion of $\vec{L}$.

In case $A = B$, solve them and compute $L_1 = L_1(t)$, $L_2 = L_2(t)$, $L_3 = L_3(t)$ explicitly as functions of time t.

B. QUANTUM MECHANICS

Consider a system of two spin-1/2 particles, interacting with an external magnetic field along the z axis and among themselves. The Hamiltonian is

$$H = 4a\vec{s}_1 \cdot \vec{s}_2 + 2b(s_{1z} + s_{2z})\,. \tag{3}$$

Taking $\hbar = 1$ for simplicity, represent as usual the two spins as

$$s_{1i} = \frac{1}{2}\sigma_i \otimes 1_2\,, \qquad s_{2i} = \frac{1}{2}1_2 \otimes \sigma_i\,, \tag{4}$$

where σ_i are the Pauli matrices and 1_2 is the 2×2 identity.

a) At time $t = 0$, we measure both spins in the $\hat{x}$ direction and we get 1/2 for both. Write down the corresponding state ψ_0.

b) Write down the matrix form of H, specifying what basis states you are using.

c) Write down the projectors P_s on the singlet and P_t on the space of triplet states. Show that P_s commutes with H; in other words, H doesn't mix the singlet $|\mathrm{s}\rangle$ with the triplet. Evolving ψ_0, what is the probability that at time t we measure the state to be $|\mathrm{s}\rangle$?

d) Evolving ψ_0, at a time t we measure both spins along $\hat{z}$. What is the probability of finding 1/2 for both?

e) Suppose now we evolve ψ_0, and at a time T we measure the first spin along $\hat{z}$, finding 1/2. We then evolve the resulting state for a further time T, and we measure the second spin along $\hat{z}$. What is the probability of finding 1/2?

C. COMPLEX ANALYSIS

a) Consider the sequence of polynomials $p_n(z) := z^n + 2z^3 + 2z + 4$. Show that any sequence of mutually distinct roots of the collection $\{p_n : n \in \mathbb{N}\}$ converges in modulus to the unit circle $|z| = 1$.

b) Find all functions $f(z)$ satisfying:

1. $f(z)$ is analytic on $\{\Im(z) > 0\}$;
2. $f(z)$ is continuous on $\{\Im(z) \geq 0\}$;
3. $f(z)$ is real–valued on the real axis;
4. $|f(z)| > |\sin(z)|$ on $\{\Im(z) > 0\}$.

D. DIFFERENTIAL GEOMETRY

Fix $r > 0$ and let $S = \{x^2 + y^2 + z^2 = r^2\} \subset \mathbb{R}^3$ denote the sphere of radius r endowed with the metric induced from $\mathbb{R}^3$.

(a) Compute the geodesic curvature $k_g(h)$ of the curve γ_h on S given by the equation $z = h$ for $0 \leq h < r$. Is $k_g(h)$ bounded? Is its ratio with the curvature of γ_h thought of as a curve in $\mathbb{R}^3$ bounded?

(b) Suppose $\gamma_h(s)$ is a simple parametrisation by arc length of γ_h and let $X(s)$ denote a parallel unit vector field along $\gamma_h(s)$. Show that the infinitesimal change, with respect to s, of the angle formed by $X(s)$ and the curve $\gamma_h(s)$ equals the geodesic curvature $k_g(s)$.

(c) Let $0 \leq \theta(h) \leq \pi$ denote the *convex* angle formed by $X(0)$ and its parallel transport along $\gamma_h(s)$ after a time equal to the length of $\gamma_h(s)$. Show that $\theta(h)$ is well defined and achieves its maximum θ_M for a unique $\bar{h} \in (0, r)$. Compute θ_M and $\bar{h}$.

E. ALGEBRAIC GEOMETRY

Let k be an algebraically closed field.

a) Given a polynomial $f \in k[x_1, \ldots, x_n]$, prove that it determines a homogeneous polynomial $f^{\text{hom}} \in k[x_0, x_1, \ldots, x_n]$.

If $I \subset k[x_1, \dots, x_n]$ is an ideal, let I^{hom} be the ideal of $k[x_0, x_1, \dots, x_n]$ generated by the set

$$\{f^{\text{hom}} \text{ for } f \in I\}.$$

b) Let $X \subset \mathbb{A}^n_k$ be an affine variety determined by an ideal $I \subset k[x_1, \dots, x_n]$. Prove that the closure $\bar{X}$ of X in $\mathbb{P}^n_k$ is the projective variety $V(I^{\text{hom}})$.

c) Let $X \subset \mathbb{A}^2_k$ be the affine variety determined by the ideal

$$I = (f, g) \subset k[x, y] \quad \text{with} \quad f(x, y) = x, \; g(x, y) = y - x^2.$$

Determine its closure $\bar{X}$ in $\mathbb{P}^2_k$.

d) Compare $\bar{X}$ with the projective variety $V(f^{\text{hom}}, g^{\text{hom}})$.

e) Let $f_1, \dots, f_m$ be generators of an ideal $I \subset k[x_1, \dots, x_n]$. Find a condition under which

$$V(f_1^{\text{hom}}, \dots, f_m^{\text{hom}}) = V(I^{\text{hom}}).$$