S.I.S.S.A.

Sector of Functional Analysis and Applications

Entrance examination — October 10^{th}, 1994

Solve at most five of the following problems, explaining your answers.

1. Let $f \in C^\infty(\mathbf{R})$.
 (i) Prove that for any $x \in \mathbf{R}$ the Cauchy problem

$$\begin{cases} u' = e^{-f(u)^2} \\ u(0) = x \end{cases}$$

 admits a unique solution $u(\cdot, x)$ defined on $\mathbf{R}$.
 (ii) Show that $\lim_{t\to-\infty} u(t,x) = -\infty$ and $\lim_{t\to+\infty} u(t,x) = +\infty$.

2. Let

$$c_0 = \{ (x_n)_{n\in\mathbf{N}} \subset \mathbf{R} \, : \, \lim_{n\to\infty} x_n = 0 \}.$$

and let $u \, : \, c_0 \to \mathbf{R}$ be a linear bounded functional. For every $n \in \mathbf{N}$ set $\eta_n = u(e^{(n)})$, where $e^{(n)}_j = 0$ for $j \neq n$ and $e^{(n)}_n = 1$. Prove that $(\eta_n) \in \ell^1$ and $\|u\| = \sum_{n\in\mathbf{N}} |\eta_n|$.

3. Let $f \, : \, \mathbf{R} \to \mathbf{R}$ be a convex function and for every $x \in \mathbf{R}$ let

$$f'_+(x) = \lim_{h\to 0+} \frac{f(x+h) - f(x)}{h}.$$

Show that

$$\int_a^b f'_+(x)\, dx = f(b) - f(a)$$

for any $a, b \in \mathbf{R}$ with $a < b$.

4. Let (X, d) be a complete metric space, $\bar{x} \in X$, $r > 0$ and $D = \{ x \in X \, : \, d(x, \bar{x}) \leq r \}$. Let $f \, : \, D \to X$ satisfying

$$d(f(x), f(y)) \leq k\, d(x,y) \qquad \text{for any } x, y \in D$$

where $k \in (0,1)$ is a constant. Prove that if $d(\bar{x}, f(\bar{x})) \leq r(1-k)$ then f admits a unique fixed point.

5. Let $I = (0,1)$ and let (f_n) be a bounded sequence in $L^p(I)$ $(1 < p < \infty)$. Prove that, if (f_n) converges to 0 in $L^1(I)$, then it converges to 0 in $L^r(I)$ for any r with $1 \le r < p$.

6. Characterize the points of $\mathbf{R}^2$ where the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$f(x,y) = \max\{|x|,\ |y|\}$$

is differentiable.

7. For any $\alpha \ge 0$ let X^α be the space of all functions $u \in L^2(0,2\pi)$ whose Fourier coefficients

$$u_n = \int_0^{2\pi} \frac{e^{-int}}{\sqrt{2\pi}} u(t)\, dt$$

satisfy

$$\sum_{n\in\mathbf{Z}} |n|^{2\alpha} |u_n|^2 < \infty.$$

It is known that X^α is a Hilbert space with norm

$$\|u\|_\alpha = \left(|u_0|^2 + \sum_{n\in\mathbf{Z}} |n|^{2\alpha} |u_n|^2 \right)^{\frac{1}{2}}.$$

Prove that, if $0 \le \alpha < \beta$, then the imbedding $X^\beta \hookrightarrow X^\alpha$ is compact.

8. Let $B = \{ (x,y) \in \mathbf{R}^2 \ : \ x^2 + y^2 < 1 \}$. Find the solution $u \in C^2(B) \cap C^0(\bar{B})$ to the problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + \dfrac{\partial^2 u}{\partial y^2}(x,y) = 0 & \forall (x,y) \in B \\ u(x,y) = 2x^2 & \forall (x,y) \in \partial B. \end{cases}$$

9. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. Prove that f is convex if and only if

$$\int_{-\infty}^{+\infty} f(x)[\varphi(x+h) + \varphi(x-h) - 2\varphi(x)]\, dx \ge 0$$

for any $h \in \mathbf{R}$ and for any $\varphi \in C_0^\infty(\mathbf{R})$ with $\varphi \ge 0$.

10. Let $\mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ be the space of all linear maps from $\mathbf{R}^n$ into $\mathbf{R}^n$, let $A : \mathbf{R}^n \to \mathcal{L}(\mathbf{R}^n, \mathbf{R}^n)$ be a function of class C^1 and let $f(x) = A(x)\,x$ for every $x \in \mathbf{R}^n$. Show that if $A(0)$ is invertible then f is locally invertible in a neighborhood of 0.